The Fuzzy Logic of Prudence and Caution*

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Eubouliatic logic

Alan Ross Anderson (1968) was interested in the logic of prudence and related notions, such as caution.

He called this logic "eubouliatic logic" (from the Greek *euboulos*, meaning "prudent"). But this name is misleading, if only because *euboulia* ("deliberating well") is an activity (Aquinas, 1265–74, q. 51 art. 2), whereas being prudent is a property or quality. Anderson was hardly interested in *euboulia*, but he was highly interested in being prudent.

We will use the term "prudence-related" instead of "eubouliatic" and "logic of prudence" instead of "eubouliatic logic."

Logic of prudence

Anderson defined his logic of prudence $\mathbf{E_R}$ as relevant system $\mathbf{R}$ plus:

- Constant $\mathcal{G}$ ("the good thing").
- Operator $\mathcal{P}$ ("it is prudent that").
- Definition $\mathcal{P}A \stackrel{\text{def}}{=} A \to \mathcal{G}$ ("A implies ["ensures," "guarantees"] the good thing").
- Axiom $\neg(\neg\mathcal{G} \to \mathcal{G})$ ("axiom of avoidance").

As usual, $\neg A \stackrel{\text{def}}{=} A \to f$, $A \leftrightarrow B \stackrel{\text{def}}{=} (A \to B) \wedge (B \to A)$, $A \circ B \stackrel{\text{def}}{=} \neg(A \to \neg B)$.
A *relevant* logic is a logic in which $A \to B$ is a theorem if and only if (i) A and B share a propositional variable or (meta-definable) propositional constant and (ii) either $\neg A$ or B is not a theorem.

Prudence-related fragment

The prudence-related fragment of $\mathbf{E_R}$ ($\mathbf{E_R}$ without $\mathcal{G}$) can be axiomatized as $\mathbf{R}$ plus the following axioms (Lokhorst, 2008):

(E1) $(A \to B) \to (\mathcal{P}B \to \mathcal{P}A)$.

(E2) $A \to \mathcal{P}\mathcal{P}A$.

(E3) $\mathcal{P}A \to \neg\mathcal{P}\neg A$.

Proof: for each derivation $A_1, \ldots, A_n$ define $\mathcal{G}$ as $\mathcal{P}t$, where $t \stackrel{\text{def}}{=} \bigwedge_{i=1}^{m}(p_i \to p_i)$ and p_1, ..., p_m are the propositional variables occurring in $A_1, \ldots, A_n$.

Additional notions

Some additional notions can be defined as follows:

1. $\mathcal{P}_w A$ ("it is imprudent that A"): A is not prudent:
   $\mathcal{P}_w A \stackrel{\text{def}}{=} \neg \mathcal{P} A \stackrel{\text{def}}{=} \neg(A \to \mathcal{G})$.
2. $\mathcal{C} A$ ("it is cautious that A"): $\neg A$ is not prudent:
   $\mathcal{C} A \stackrel{\text{def}}{=} \neg \mathcal{P} \neg A \stackrel{\text{def}}{=} \neg(\neg A \to \mathcal{G})$.
3. $\mathcal{C}_w A$ ("it is incautious that A"): $\neg A$ is prudent:
   $\mathcal{C}_w A \stackrel{\text{def}}{=} \mathcal{P} \neg A \stackrel{\text{def}}{=} \neg A \to \mathcal{G}$.

The same notion of caution plays a role in the "precautionary principle" (Ewald, Gollier, de Sadeleer, 2001).

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Alternative axiomatizations

$\mathbf{E_R}$ can alternatively be axiomatized as follows (Lokhorst, 2008):

1. **R** plus $(A \to B) \to (\mathcal{P}_w A \to \mathcal{P}_w B)$, $\mathcal{P}_w A \to (A \circ \mathcal{P}_w t)$, $\neg\mathcal{P}_w A \to \mathcal{P}_w \neg A$, $\mathcal{P}A \leftrightarrow \neg\mathcal{P}_w A$.

   $(A \to B) \to (\neg\mathcal{P}A \to \neg\mathcal{P}B)$, $\neg\mathcal{P}A \to \neg(A \to \neg\neg\mathcal{P}t)$, $\neg\neg\mathcal{P}A \to \neg\mathcal{P}\neg A$, $\mathcal{P}A \leftrightarrow \neg\neg\mathcal{P}A$.

2. **R** plus $(A \to B) \to (\mathcal{C}B \to \mathcal{C}A)$, $\mathcal{C}\mathcal{C}A \to A$, $\neg\mathcal{C}A \to \mathcal{C}\neg A$, $\mathcal{P}A \leftrightarrow \neg\mathcal{C}\neg A$.

   $(A \to B) \to (\neg\mathcal{P}\neg B \to \neg\mathcal{P}\neg A)$, $\neg\mathcal{P}\neg\neg\mathcal{P}\neg A \to A$ and $\neg\neg\mathcal{P}\neg A \to \neg\mathcal{P}\neg\neg A$, $\mathcal{P}A \leftrightarrow \neg\neg\mathcal{P}\neg\neg A$.

3. **R** plus $(A \to B) \to (\mathcal{C}_w A \to \mathcal{C}_w B)$, $\mathcal{C}_w(\mathcal{C}_w A \to A)$, $\mathcal{C}_w A \to \neg\mathcal{C}_w \neg A$, $\mathcal{P}A \leftrightarrow \mathcal{C}_w \neg A$.

   $(A \to B) \to (\mathcal{P}\neg A \to \mathcal{P}\neg B)$, $\mathcal{P}\neg(\mathcal{P}\neg A \to A)$, $\mathcal{P}\neg A \to \neg\mathcal{P}\neg\neg A$, $\mathcal{P}A \leftrightarrow \mathcal{P}\neg\neg A$.

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Square of opposition

1. $\vdash \mathcal{P}_w A \leftrightarrow \neg \mathcal{P} A$: $\mathcal{P} A$ and $\mathcal{P}_w A$ are contradictories.
2. $\vdash \mathcal{C}_w A \leftrightarrow \neg \mathcal{C} A$: $\mathcal{C} A$ and $\mathcal{C}_w A$ are contradictories.
3. $\vdash \mathcal{P} A \rightarrow \mathcal{C} A$, which agrees with Webster (1828, "prudence") and Aquinas (1265–74, q. 49 art. 8): $\mathcal{P} A$ and $\mathcal{C} A$ are subalterns.
4. $\vdash \mathcal{C}_w A \rightarrow \mathcal{P}_w A$: $\mathcal{C}_w A$ and $\mathcal{P}_w A$ are subalterns.
5. $\vdash \neg(\mathcal{P} A \wedge \mathcal{C}_w A)$: $\mathcal{P} A$ and $\mathcal{C}_w A$ are contraries.
6. $\vdash \mathcal{C} A \vee \mathcal{P}_w A$: $\mathcal{C} A$ and $\mathcal{P}_w A$ are subcontraries.

These notions can be depicted in a square of opposition (Fig. 1).

Illustration 1

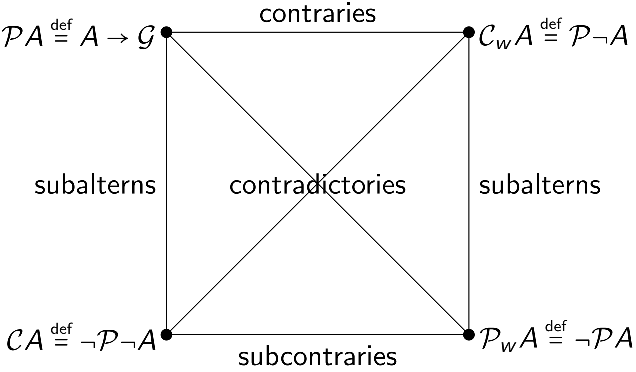

Figure: Anderson 1968, Fig. 8.

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Illustration 2

This square of opposition is the same as the square of opposition of Apuleius of Madaura. The A E I O propositions are familiar from the medieval Aristotle tradition.

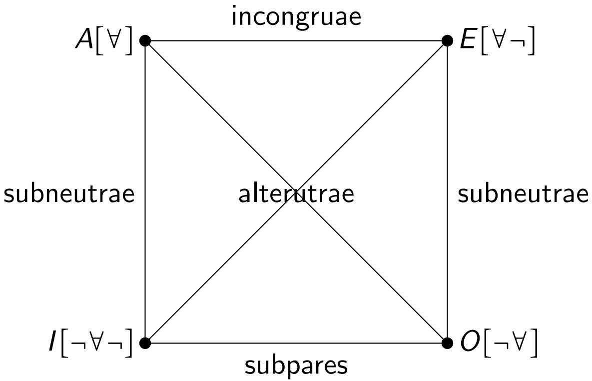

Figure: First-order predicate calculus plus $\forall x F x \rightarrow \exists x F x$. Apuleius 300, Gombocz 1990, Anderson 1968, Fig. 2.

Illustration 3

This diagram is better known in the version of Boethius:

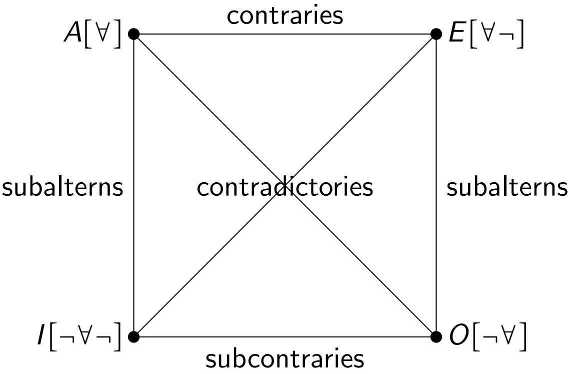

Figure: First-order predicate calculus plus $\forall x Fx \to \exists x Fx$. Boethius 500, Anderson 1968, Fig. 2.

Illustration 4

The modal square of opposition is similar:

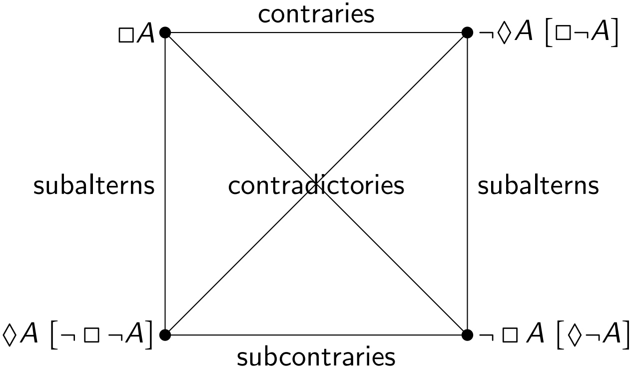

Figure: Modal system **KD**. Boethius 500, Anderson 1968, Fig. 3.

Illustration 5

The deontic square of opposition is also similar:

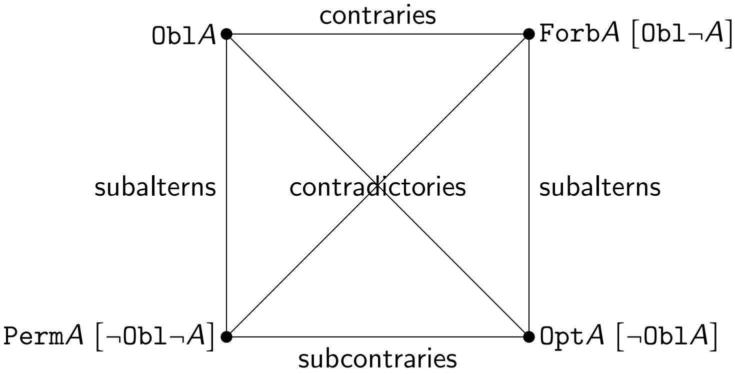

Figure: Standard deontic logic. Leibniz (Lenzen 2005), Anderson 1968, Fig. 4.

Claim

The square of opposition drawn in Fig. 1 is an acceptable representation of the relations obtaining between these concepts.

Anderson's account was slightly different from ours because he thought that $\mathcal{P}A$ could also be read as "it is safe that A." As we have argued elsewhere (Lokhorst, 2008), "it is safe that A" should be represented as $\mathcal{C}A$ rather than $\mathcal{P}A$.

Modification: Fuzzification

Basic idea: “prudence” and “the good thing” are *fuzzy concepts*.

Hedges

This is illustrated by the concept of “hedges.” The operator “prudence” is typically used with linguistic “hedges”: sort of, kind of, loosely speaking, more or less, on the ... side, roughly, pretty (much), relatively, somewhat, rather, mostly, technically, strictly speaking, essentially, in essence, basically, particularly, par excellence, largely, for the most part, very, highly, especially, exceptionally, quintessentially, literally, often, almost, typically/typical, as it were, in a sense, in one sense, in a real sense, in an important sense, in a way, details aside, so to say, practically, anything but, nominally, in name only, actually, really, ... (Lakoff, 1973).

Solution

These hedges imply that "prudence" *itself* is a fuzzy concept.

Hedges for crisp, black/white concepts simply do not make sense.

This explains why there are so many jokes about women who are "a little bit pregnant" (and not about women who are "somewhat prudent").

Fuzzy relevant logic

A *fuzzy* logic is a logic in which there are not just two extreme degrees of truth—falsity (0) and truth (1)—but various intermediate degrees of truth, i.e., degrees of truth between 0 and 1. These degrees of truth are linearly ordered, so that every two values are comparable, i.e., for any A, B either $v(A) \leq v(B)$ or $v(B) \leq v(A)$ (Cintula, 2006).

The oldest example of a fuzzy logic is Łukasiewicz's logic **Ł3**, in which sentences have values 0, $\frac{1}{2}$ or 1 and $0 < \frac{1}{2} < 1$.

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RM

There is one well-known logic that is fuzzy and "semi-relevant": **RM**, which is **R** plus axiom scheme $A \to (A \to A)$. **RM** has theorem $(A \to B) \vee (B \to A)$, so it is fuzzy.

However, **RM** also has theorems

(i) $(A \wedge \neg A) \to (B \vee \neg B)$, $\neg(A \wedge \neg A)$ and $B \vee \neg B$; and

(ii) $\neg(A \to A) \to (B \to B)$, $\neg\neg(A \to A)$ and $B \to B$;

so **RM** is not relevant.

FR

Fuzzy relevant logic **FR** (Metcalfe and Montagna, 2007) (Yang, 2012) (Yang, 2014) (Yang, 2015) is **R** plus:

(Lin) $(A \to B) \vee (B \to A)$.

Theorem

R $\subset$ **FR** $\subset$ **RM**.

$\mathbf{R} \subset \mathbf{FR} \subset \mathbf{RM}$

- **R** is weaker than **FR** because $((A \to (B \vee C)) \wedge (B \to C)) \to (A \to C)$ is provable in **FR** but not in **R** (Yang, 2012, (5)).
- **FR** is weaker than **RM** because $A \to (A \to A)$ is provable in **RM** but not in **FR**, as shown below (matrix generated by `MaGIC` (Slaney, 2008)).

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Matrix

Logic: **R**. Extra: $(A \to B) \vee (B \to A)$.
Fragment: $\to$, $\wedge$, $\vee$, $\neg$, $\circ$, t, f, T, F.
Fail: $A \to (A \to A)$. Negation table:

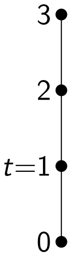
a	0	1	2	3
$\neg a$	3	2	1	0

Order: $0 \leq 1 \leq 2 \leq 3$. Choice of t: 1.

Implication matrix:

$\to$	0	1	2	3
0	3	3	3	3
1	0	1	2	3
2	0	0	1	3
3	0	0	0	3

Failure: $2 \to (2 \to 2)$.



$\mathcal{P}p$

Theorem

RM *provides* $\mathcal{P}p$, **FR** *does not.*

$\vdash \mathcal{P}p$ iff $\vdash p \to \mathcal{G}$ iff $\vdash p \to (p \to p)$ by the definition of $\mathcal{G}$ (in the context of $p \to \mathcal{G}$) given above. $p \to (p \to p)$ is a theorem of **RM**, but not a theorem of **FR**.

R: Algebraic semantics

A *De Morgan monoid* (Dunn-algebra) is a structure $\mathbf{A} = (A, t, f, \wedge, \vee, *, \to)$, where

1. $(A, \wedge, \vee)$ is a distributive lattice,
2. $(A, *, t)$ is a commutative monoid,
3. $y \le x \to z$ iff $x * y \le x$, for all $x, y, z \in A$ (residuation),
4. $x \le x * x$ (contraction),
5. $((x \to f) \to f) \le x$ (double negation elimination).

R: Evaluations

Let $\mathcal{A}$ be a De Morgan monoid. An $\mathcal{A}$-evaluation is a function $v\colon \mathrm{WFF} \to \mathcal{A}$ satisfying

1. $v(A \to B) = v(A) \to v(B)$,
2. $v(A \wedge B) = v(A) \wedge v(B)$,
3. $v(A \vee B) = v(A) \vee v(B)$,
4. $v(A \circ B) = v(A) * v(B)$,
5. $v(f) = f$.

A is $\mathcal{A}$-valid iff $t \leq v(A)$ for all $\mathcal{A}$-evaluations v. An $\mathcal{A}$-model of T is an $\mathcal{A}$-evaluation such that $t \leq v(A)$ for all $A \in T$. $\mathrm{Mod}(T, \mathcal{A})$ is the class of $\mathcal{A}$-models of T. A is a semantic consequence of T with respect to $\mathcal{K}$ iff $\mathrm{Mod}(T, \mathcal{A}) = \mathrm{Mod}(T \cup \{A\}, \mathcal{A})$ for all $\mathcal{A} \in \mathcal{K}$.

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R: Soundness and strong completeness

We write $T \vdash A$ for A is derivable from T. Note that $T \cup \{A\} \vdash B$ iff $T \vdash (A \wedge t) \to B$. Let T be a theory over **R**. Let $[A]_T = \{B \in \text{WFF}\colon T \vdash A \leftrightarrow B\}$. A_T is the set of all classes $[A]_T$. A_T is a De Morgan monoid.

Theorem

$T \vdash A$ iff A is a semantic consequence of T with respect to the class of De Morgan monoids.

(**R** is sound and strongly complete with respect to the class of De Morgan monoids.)

FR: Algebraic semantics

Eunsuk Yang (Yang, 2012, 2014).
A *chain* is a De Morgan monoid satisfying $x \leq y$ or $y \leq x$ (linear order).

Theorem
$T \vdash_{\mathbf{FR}} A$ iff A is a semantic consequence of T with respect to the class of chains.

(**FR** is sound and strongly complete with respect to the class of chains.)

FR: Algebraic Kripke-style semantics

Eunsuk Yang (Yang, 2012, 2014).
A *frame* is a structure $\mathbf{X} = \langle X, t, f, \leq, *, \rightarrow \rangle$ such that $\langle X, t, f, \leq, *, \rightarrow \rangle$ is a linearly ordered residuated pointed commutative monoid satisfying $x = (x \rightarrow f) \rightarrow f$ and $x \leq x * x$.
The members of X are called *nodes*.

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FR: Forcing

A *forcing* is a relation between nodes and propositional variables such that:

1. if $p \in \mathrm{AT}$ then if $x \vDash p$ and $y \leq x$, then $y \vDash p$ (backward heredity),
2. $x \vDash t$ iff $x \leq t$,
3. $x \vDash f$ iff $x \leq f$,
4. $x \vDash A \wedge B$ iff $x \vDash A$ and $x \vDash B$,
5. $x \vDash A \vee B$ iff $x \vDash A$ or $x \vDash B$,
6. $x \vDash A \circ B$ iff there are $y, z \in X$ such that $y \vDash A$, $z \vDash B$ and $x \leq y * z$,
7. $x \vDash A \rightarrow B$ iff for all $y \in X$, if $y \vDash A$, then $x * y \vDash B$.

FR: Soundness

A *model* is a pair $(\mathbf{X}, \vDash)$ where $\mathbf{X}$ is a frame and $\vDash$ is a forcing on X. A is *true* in $(\mathbf{X}, \vDash)$ iff $t \vDash A$. A is *valid* in $\mathbf{X}$ ($\mathbf{X} \vDash A$) iff A is true in $(\mathbf{X}, \vDash)$ for every forcing $\vDash$ on $\mathbf{X}$.

Theorem

If $\vdash_{\mathbf{FR}} A$, then A is valid in all frames.

FR: Strong completeness

1. The $\{t, f, \leq, *, \rightarrow\}$ reduct of a chain is a frame.
2. Let $\mathbf{X} = \langle X, t, f, \leq, *, \rightarrow\rangle$ be a frame. Then $\mathbf{A} = \langle X, t, f, \max, \min, *, \rightarrow\rangle$ is a chain (where max and min are meant with respect to $\leq$).
3. Let $\mathbf{X}$ be the $\{t, f, \leq, *, \rightarrow\}$ reduct of a chain $\mathbf{A}$ and let v be an evaluation in $\mathbf{A}$. Let $x \vDash p$ iff $x \leq v(p)$ for all $x \in \mathbf{A}$ and all $p \in \mathrm{AT}$. Then $(\mathbf{X}, \vDash)$ is a model and $x \vDash A$ iff $x \leq v(A)$.

Theorem

If $T \vDash A$, then $T \vdash_{\mathbf{FR}} A$.

(**FR** is strongly complete with respect to the class of frames.)

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Fuzzy logic of prudence

Fuzzy logic of prudence $\mathbf{E_{FR}}$ is $\mathbf{FR}$ plus the axiom of avoidance and the four operators $\mathcal{P}$, $\mathcal{P}_w$, $\mathcal{C}$, $\mathcal{C}_w$, defined as in $\mathbf{E_R}$.

$\mathbf{E_{FR}}$ extends $\mathbf{E_R}$, so all results obtained above for $\mathcal{P}$, $\mathcal{P}_w$, $\mathcal{C}$, $\mathcal{C}_w$ in $\mathbf{E_R}$ also hold in $\mathbf{E_{FR}}$.

Conclusion

Anderson's logic of prudence can be extended to a fuzzy logic of prudence. This does not affect the prudence-related square of opposition.

Anderson remarked that he was "far from satisfied with [his] terminological choices." In contrast to Anderson, we are completely satisfied with our terminological choices, mainly because (i) we have avoided the term "eubouliatic" and (ii) we have identified safety with caution rather than prudence.

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