MULTIPLY MODAL EXTENSIONS OF DA COSTA'S $\mathscr{C}_n$, $1 \leq n \leq \omega$, LOGICAL RELATIVISM, AND THE IMAGINARY

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Abstract. How should *our* logic express what *other* logics deem necessary? How should we give a rational account of forms of rationality which are different from ours? The present paper answers these questions. It shows how to enrich logical systems with operators which describe what is necessary, rational and imaginary according to other systems. Although only da Costa's paraconsistent calculi are treated in detail, the construction is generally applicable. As a result the thesis of logical relativism—people from different cultures may live in different cognizable worlds—may henceforth be discussed in terms of modal logic and possible world semantics.

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Or, s'il y a plusieurs mondes, comme ... presque toute la philosophie a pensé, que sçavons nous si les principes et les règles de cettuy touchent pareillement les autres? Ils ont à l'avanture autre visage et autre police.

M. E. de Montaigne, *Apologie de Raimond Sebond*
(de Montaigne 1580)

1. Introduction

When one does not restrict one's attention to just one logic, but bears in mind that there is a plurality of logics around (intuitionistic logic, multi-valued logics, paraconsistent logics, etc.), it seems obvious that notions such as necessity, possibility and rationality are not *absolute*, but *relative* to the particular logical system under consideration. Yet, the logic-relative nature of these notions is not generally recognized, and logical systems which take it into account do not seem to have been constructed up to now. In this paper, we will try to fill this lacuna. We have selected da Costa's well-known series of paraconsistent logics $\mathscr{C}_n$, $0 \leq n \leq \omega$, to make a first study of logic-relativized notions of necessity, possibility and rationality, and to indicate some philosophical areas (Vasil'ev's "imaginary logic", the logic of belief, and Lévy-Bruhl's "logical relativism") which are illuminated by the relativistic and pluralistic analysis of these notions.

The general considerations motivating our enterprise are as follows.

In ordinary single-operator modal logic, the sentence "it is logically necessary that A" is given the following truth-condition.

"It is logically necessary that A" is true at world w (in model $\mathfrak{M}$) iff "A" is true at all logically possible worlds accessible from w (in $\mathfrak{M}$).

But now suppose we consider several logical systems at once, say the da Costa series $\mathscr{C}_n$, $0 \leq n \leq \omega$. In this case, the above truth-condition can no longer be used. For to *which* one of the many systems does "logically necessary" refer now? Any system from the da Costa series may be meant. And according to *which* one of the various logics are the worlds referred to logically possible? Again, any system from the da Costa series may be meant.

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To remove this ambiguity, one has to specify which logic one has in mind in both cases. This is what we will do in the following. Each of the systems we present has denumerably many modal operators $\Box_n$, $0 \leq n \leq \omega$, corresponding to $\mathscr{C}_n$, $0 \leq n \leq \omega$, respectively. The subscripts of the operators indicate the logical systems to which they are relativized; $\Box_n A$ may be read as "it is $\mathscr{C}_n$-necessary that A", or as "according to $\mathscr{C}_n$, it is necessary that A". Semantically, we introduce various sets of worlds W_n, $0 \leq n \leq \omega$, likewise corresponding to $\mathscr{C}_n$, $0 \leq n \leq \omega$, respectively; W_n is the set of worlds which are possible according to logic $\mathscr{C}_n$, or stated otherwise, it is the set of worlds in which $\mathscr{C}_n$ is valid. Having made these distinctions, we are able to give the following disambiguated truth-condition:

> "It is $\mathscr{C}_n$-necessary that A" is true at w (in $\mathfrak{M}$) iff "A" is true at all $\mathscr{C}_n$-possible worlds accessible from w (in $\mathfrak{M}$).

In this way, we explicitly recognize the fact that there is more than one logic around. The truth-condition has the effect that $\mathscr{C}_n$-axioms are $\mathscr{C}_n$-necessary but need not be $\mathscr{C}_m$-necessary if $m \neq n$, which is in accord with our intuitions on logic-relative necessity. Since $\mathscr{C}_n \subseteq \mathscr{C}_m$ if $n \geq m$, we will make the plausible assumption that $W_m \subseteq W_n$ if $m \leq n$ for all m, $0 \leq m \leq \omega$, n, $0 \leq n \leq \omega$ (the stronger the logic, the less it will count as possible). Thus, $\mathscr{C}_n$-axioms are $\mathscr{C}_m$-necessary if $n \geq m$.

The series of the smallest logical systems arising from this semantic condition will be denoted as $\mathscr{C}_nK$, $0 \leq n \leq \omega$. We will study this series and some related ones in §2 below. The applications of the systems will be discussed in §3.

First, however, a preliminary remark. As we have said, it is our goal to apply our systems to the analysis of Vasil'ev's views. Now according to Arruda (1977), an imaginary logic in the sense of Vasil'ev must be adequate to handle at least two sorts of negation, viz., classical (strong) negation, and a weaker negation for which the law of contradiction is not valid. The former type of negation may be defined in $\mathscr{C}_n$, $0 \leq n < \omega$; however, it may *not* be defined in $\mathscr{C}_\omega$ (Arruda mistakenly claims the contrary). In order to remove this difficulty, we have enriched the language with a primitive symbol for strong negation, notated as $\approx$. This has no effect on $\mathscr{C}_n$, $0 \leq n < \omega$, except that conjunction and disjunction may now be defined in terms of $\supset$ and $\approx$. However, it makes our $\mathscr{C}_\omega$ stronger than da Costa's $\mathscr{C}_\omega$. For example, Peirce's law $(((A \supset B) \supset A) \supset A)$ may now be proven in $\mathscr{C}_\omega$ (in the same way as in classical logic), which is impossible in da Costa's original $\mathscr{C}_\omega$.[1]

2. The series $\mathscr{C}_nK$, $0 \leq n \leq \omega$

2.1. **The language.** Let AT be a denumerable set. The set of formulas WFF is the smallest set such that AT $\subseteq$ WFF and if A, $B \in$ WFF then $\sim A$, $\approx A$, $A \supset B$, $\Box_n A \in$ WFF, for each n, $0 \leq n \leq \omega$.

Definitions: $A \,\&\, B$, $A \vee B$ and $A \equiv B$ are defined as usual. A^o is short for $\sim(A \,\&\, \sim A)$. A^n is short for $A^{\overbrace{oo\ldots o}^{n}}$, i.e., for A followed n times by ^{o}. $A^{(n)}$ abbreviates $\overbrace{A^o \,\&\, A^{oo} \,\&\, \ldots \,\&\, A^n}^{n}$. Finally, $\sim^{(n)} A$ stands for $\sim A \,\&\, A^{(n)}$, and $\Diamond_n A$ for $\approx \Box_n \approx A$.

2.2. **Axiomatization of $\mathscr{C}_nK$, $0 \leq n \leq \omega$.** Each $\mathscr{C}_nK$, $0 \leq n \leq \omega$, is axiomatized by adding the following axiom schemes C1–C7 and rule scheme C8 to classical propositional logic (formulated with classical negation, $\approx$):[2]

C1 $A \vee \sim A$.

[1] See da Costa (1974) and Loparić (1977) on the indefinability of classical negation and the unprovability of Peirce's theorem in $\mathscr{C}_\omega$.

[2] Cf. da Costa (1974).

C2 $\sim\sim A \supset A$.
C3 $\sim A \supset (A \supset B)$, provided that $n = 0$.
C4 $B^{(n)} \supset ((A \supset B) \supset ((A \supset \sim B) \supset \sim A))$, provided that $n \neq \omega$.
C5 $(A^{(n)} \,\&\, B^{(n)}) \supset ((A \supset B)^{(n)} \,\&\, (A \,\&\, B)^{(n)} \,\&\, (A \vee B)^{(n)})$, provided that $n \neq \omega$.
C6 $\Box_m A \supset \Box_k A$ if $m \geq k$, for each k, $0 \leq k \leq \omega$, m, $0 \leq m \leq \omega$.
C7 $\Box_m(A \supset B) \supset (\Box_m A \supset \Box_m B)$, for each m, $0 \leq m \leq \omega$.
C8 $\vdash_m A \Rightarrow \vdash_n \Box_m A$, for each m, $0 \leq m \leq \omega$.

Here $\vdash_n A$ is an abbreviation for $\emptyset \vdash_n A$, where $S \vdash_n A$ (for $S \subseteq \mathrm{WFF}$) in turn means that A is derivable from S by means of the axioms and rules of $\mathscr{C}_n K$. Derivability is defined in the usual way.

The distinction between the constant n (of $\mathscr{C}_n K$) and the variables k and m should be especially noted in the above. Furthermore, notice the special form of the denumerably many rules of necessitation (C8). In conjunction with C7, these rules have the consequence that for each sequence Σ of modal operators (including the null-sequence) and each k and m, $\{A : \vdash_k \Sigma\Box_m A\}$ is a $\mathscr{C}_m K$-theory. (A $\mathscr{C}_m K$-theory is a set of sentences containing $\mathscr{C}_m K$ and closed under modus ponens.) This justifies our reading of $\Box_m A$ as "it is $\mathscr{C}_m K$-necessary that A". Following the common doxastic interpretation of modal logic as the logic of rational belief, it allows us to read $\Box_m A$ as "it is $\mathscr{C}_m K$-rational to believe that A" or "a perfect $\mathscr{C}_m K$-logician (adherent of $\mathscr{C}_m K$) believes that A". (See §3.2 below.)

It may be observed that, for any m, $0 \leq m \leq \omega$, the modal fragments $\{A : \vdash_n \Box_m A\}$ of all $\mathscr{C}_n K$ are exactly the same. Furthermore, the $\{\approx, \supset, \Box_m\}$ fragment of each $\mathscr{C}_n K$ is exactly the same as the classical modal system K.

Finally, notice that $\mathscr{C}_n K \subseteq \mathscr{C}_m K$ if $n \geq m$. The strongest logic is $\mathscr{C}_0 K$, while $\mathscr{C}_\omega K$ is the weakest one. In $\mathscr{C}_n K$, $0 \leq n < \omega$, strong negation may be defined as $\approx A = \sim^{(n)} A$. In $\mathscr{C}_\omega K$ it cannot be but primitive.

2.3. **Semantics.** A Kripke-style "possible-worlds" model for $\mathscr{C}_n K$ is a structure

$$\mathfrak{M} = \langle \langle W_m \rangle_{0 \leq m \leq \omega}, W_n, w_0, R, V \rangle,$$

where:

- each W_m, $0 \leq m \leq \omega$, is a set (of $\mathscr{C}_m K$-possible worlds);
- $W_k \subseteq W_m$ if $k \leq m$;
- W_n is the distinguished set of "really possible" (i.e., $\mathscr{C}_n K$-possible) worlds;
- w_0 (the actual world) is a member of W_n;
- $R \subseteq W \times W$, where $W = \bigcup\{W_m : 0 \leq m \leq \omega\} = W_\omega$;
- $V : \mathrm{WFF} \times W \mapsto \{0, 1\}$ is a function satisfying the following conditions:[3]
    1. $V(\approx A, w) = 1$ iff $V(A, w) = 0$;
    2. $V((A \supset B), w) = 1$ iff $V(A, w) = 0$ or $V(B, w) = 1$;
    3. if $V(\sim A, w) = 0$ then $V(A, w) = 1$;
    4. if $V(\sim\sim A, w) = 1$ then $V(A, w) = 1$;
    5. if $V(\sim \mathrm{A}, w) = 1$ then $V(A, w) = 0$, provided that $w \in W_0$;
    6. For all m, $0 \leq m < \omega$: if $V(B^{(m)}, w) = V((A \supset B), w) = V((A \supset \sim B), w) = 1$ then $V(A, w) = 0$, provided that $w \in W_m$;
    7. For all m, $0 \leq m < \omega$: if $V(A^{(m)} \,\&\, B^{(m)}, w) = 1$ then $V(((A \supset B)^{(m)} \,\&\, (A \,\&\, B)^{(m)} \,\&\, (A \vee B)^{(m)}), w) = 1$, provided that $w \in W_m$;
    8. $V(\Box_m A, w) = 1$ iff $V(A, v) = 1$ for all $v \in W_m$ such that wRv.

For $S \subseteq \mathrm{WFF}$, $S \models_n A$ means: for all $\mathscr{C}_n K$-models in the above sense: if $V(B, w_0) = 1$ for all $B \in S$, then $V(A, w_0) = 1$. For all m, $0 \leq m \leq \omega$, we say that A is *valid* on W_m (in a particular model) iff $V(A, w) = 1$ for all $w \in W_m$.

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[3] Cf. da Costa & Alves (1977) for the non-modal conditions.

2.4. **Completeness.**

Completeness theorem: $S \vdash_n A$ iff $S \models_n A$.

Proof. From left to right: trivial. From right to left: a canonical model may be constructed in the usual way. Let $w_0 \in W_n$ be a $\mathscr{C}_n K$-maximal nontrivial extension of S, let W_n be the set of $\mathscr{C}_m K$-maximal nontrivial sets of the language (this is is the only unusual part of the construction), and let $V(A, w) = 1$ iff $A \in w$. It is not difficult to show that the canonical model satisfies all conditions from §2.3 and that, for any A which is not derivable from S, $V(B, w_0) = 1$ for all $B \in S$ while $V(A, w_0) = 0$. This completes the proof.[4]

2.5. **Some correspondence results.**

Seriality: If we add the axiom $\Diamond_\omega(A \vee {\sim}A)$ to each $\mathscr{C}_n K$, $0 \leq n \leq \omega$, we obtain a series $\mathscr{C}_n K$, $0 \leq n \leq \omega$, of systems which are complete with respect to the class of serial models (i.e., models in which $\forall w \in W \exists v \in W wRv$).

Reflexivity: corresponds to adding $\Box_n A \supset A$ to each $\mathscr{C}_n K$. (Notice that $\Box_\omega A \supset A$ would be too weak and $\Box_0 A \supset A$ too strong.)

Transitivity: corresponds to adding $\Box_m A \supset \Box_\omega \Box_m A$ (for all m, $0 \leq m \leq \omega$) to each $\mathscr{C}_n K$.

Symmetry: corresponds to adding $A \supset \Box_\omega \Diamond_n A$ to each $\mathscr{C}_n K$.

2.6. **The logic of the imaginary.** In conformity with Vasil'ev's use of the term (see §3.1 below), we say that a world $w \in W$ is *imaginary* from the point of view of $\mathscr{C}_n K$ if $w \notin W_n$. So w is imaginary according to $\mathscr{C}_n K$ if w is possible according to *some* logic, but impossible according to $\mathscr{C}_n K$ itself. Imaginary worlds are the worlds "lying beyond the horizon of the logical space" of $\mathscr{C}_n K$.

Imaginariness may be expressed in the object-language by introducing a new modal operator I_n. $I_n A$ may be read as "according to $\mathscr{C}_n K$ it is imaginary that A" or as "it is $\mathscr{C}_n K$-impossible but ($\mathscr{C}_\omega$-)imaginable that A". Thus:

$$I_n A \stackrel{\text{def}}{=} {\approx}\Diamond_n A \,\&\, \Diamond_\omega A.$$

The stronger the logic, the more will be imaginary according to it. Classical logic gives the verdict "imaginary" most easily, whereas nothing is imaginary according to $\mathscr{C}_\omega K$. (So imaginariness is as logic-relative as possibility and necessity are.)

It may be of some interest to investigate what the logic of the imaginary is like all by itself. Fortunately, the answer is easy, for the case is similar to that of "purely logical (as contrasted to physical) possibility", which has been studied by Bacon (1981).

Bringing Bacon's axiomatization into line with our notation, we may axiomatize the notion of "it is imaginary according to $\mathscr{C}_n K$" by adding the following axiom schemes I1-I4 and rule schemes I5-I7 to $\mathscr{C}_\omega K$ (for all m, n, $0 \leq m \leq \omega$, $0 \leq n \leq \omega$):[5]

I1 ${\approx}I_\omega A$.
I2 $I_n A \supset I_m A$, if $n \geq m$.
I3 $(I_n A \,\&\, I_n B) \supset I_n(A \vee B)$.
I4 $({\approx}I_n(A \,\&\, B) \,\&\, I_n A \,\&\, I_n C) \supset I_n{\approx}(C \supset (A \,\&\, B))$.

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[4] One may compare the completeness proofs of classical multiply modal logics which have been given by Fitting (1969) and Rennie (1970). On modal logic, see also Chellas (1980).

[5] Bacon's operator N of purely physical necessity corresponds to our $I_n{\approx}A$. Bacon's relation of physical accessibility S corresponds to our $R \restriction W_n$, while his relation of logical accessibility R corresponds to our R. The main differences between Bacon's systems and ours are threefold. First, we have replaced Bacon's axiom N3 by his derived rule T3. (Both are easily seen to be interderivable.) Second, we have dropped the condition that R is reflexive (corresponding to Bacon's axiom N1). Third, I1 and I2 have no counterparts in Bacon's system; they are immediate consequences of our definition of I_n and of axiom C6.

I5 $\vdash_\omega A \equiv B \Rightarrow \vdash_\omega I_n(A \equiv B)$.
I6 $\vdash_\omega A \Rightarrow \vdash_\omega \approx I_n \approx A$.
I7 $\vdash_\omega A \supset B \Rightarrow \vdash_\omega I_n(A \,\&\, C) \supset (I_n B \supset I_n A)$.

Some noteworthy theorems and derived rules are (for any n, $0 \leq n \leq \omega$):

T1 $I_n(A \vee B) \supset (I_n A \vee I_n B)$.
T2 $(I_n(A \,\&\, B) \,\&\, I_n \approx A) \supset I_n B$.
T3 $\vdash_\omega A \Rightarrow \vdash_\omega I_n(A \,\&\, B) \supset I_n B$.
T4 $\vdash_\omega A \supset B \Rightarrow \vdash_\omega I_n A \supset I_n(A \,\&\, B)$.

If $R \restriction W_n$ is serial, we have $\vdash_n A \Rightarrow \vdash_\omega \approx I_n A$. If $R \restriction W_n$ is reflexive we have $\vdash_\omega A \supset \approx I_n A$.

3. Applications

3.1. **Vasil'ev's imaginary logics and worlds.** The Russian physician Vasil'ev has become famous as one of the first forerunners of paraconsistent logic.[6] His viewpoints are clarified to a great extent by our multiply modal approach.

Inspired by the existence of various imaginary (non-Euclidean) geometries, Vasil'ev envisaged the possibility of constructing a great multitude of "imaginary" logics. These logics would enable us to study a large class of "imaginary worlds" which are impossible to classical logic, but nevertheless quite well imaginable by our minds. According to Vasil'ev, Aristotelian logic is an instrument of knowledge for only a limited class of worlds, the "classical" worlds, in which, for example, the law of non-contradiction holds. However, beyond the classical worlds there is a whole range of imaginary worlds, which obey the laws of various imaginary logics. Vasil'ev did not deny the truth of classical logic: he assumed that experience has taught us that the real world we inhabit is classical. But we can imagine that it could have been otherwise. The truth of classical logic is only an empirical matter; "logic is as empirical as geometry".[7] The idea that classical logic is universally valid is an illusion created by our particular place in logical space and a lack of imagination to look beyond the classical horizon.

Vasil'ev did not give a formal development of his views. However, an attempt to do this has been made by Arruda (1977). Arruda's formalizations indeed capture some of Vasil'ev's basic insights. However, her proposals seem to have two shortcomings. First, they do not capture Vasil'ev's central idea of a *plurality* of imaginary logics, "existing", so to say, side by side; she just presented several isolated systems. And second, she did not clarify the idea of an "imaginary world" at all, let alone the idea of a *plurality* of types of imaginary worlds, each of them possible according to some different imaginary logic. Indeed, the term "imaginary world" did not even occur in her formal exposition.

Our systems do not have these shortcomings. The introduction of several modal operators, each of them corresponding to a different logic from the da Costa series, enables us to capture the idea of a plurality of logics existing side by side. As we have seen, this has even allowed us to express the notion of "imaginary according to logic $\mathscr{C}_n$" within the language. Likewise, the multitude of types of worlds in the semantics, each type corresponding to one of the da Costa logics, seems to be a fairly direct expression of Vasil'ev's idea of a multitude of worlds described by various logics. Vasil'ev's idea that the actual world is classical may be captured

[6] Vasil'ev (1912). We follow the exposition of Vasil'ev's views given by Arruda (1984). See also Żarnecka-Biały (1985) and Puga & da Costa (1988).

[7] This famous assertion of Putnam (1968) could have come straight from Vasil'ev's writings. Putnam (1968), Rescher & Brandom (1980), and various other modern authors not only share Vasil'ev's view that classical logic could be empirically false, they even claim that it has in fact been shown to be false (by quantum mechanics).

by the condition that w_0 is a member of W_0. But even if we stipulated this we should not overlook the other worlds, and consider $\mathscr{C}_0K$, rather than the classical single-operator modal system K, as the logic of the imaginable or the possible (in a wide sense).

3.2. **The logic of belief.** Apart from clarifying Vasil'ev's ideas, our systems are also interesting from the point of view of doxastic logic.[8] Classical doxastic logic (which simply is modal logic with $\Box A$ read as "the agent believes that A") has often wrestled with the problem of how to give an account of inconsistent beliefs which does not imply that *everything* is believed. This is a problem for classical doxastic logic, because it has the (doxastic variant of the) theorem $\Box(A \mathbin{\&} {\sim}A) \supset \Box B$. The usual solution is to distinguish between "implicit" inconsistencies of the form $\Box A \mathbin{\&} \Box{\sim}A$ and "explicit" inconsistencies of the form $\Box(A \mathbin{\&} {\sim}A)$ and to deny that the former imply the latter. Thus, the "belief-set" (set of believed sentences) of the agent is generally not closed under conjunction, and it may contain at least one type of inconsistencies (implicit inconsistencies) without collapsing into the whole language.[9] Now this method of fragmentation or compartmentalisation may certainly be applicable in a number of instances, although it may sometimes have the drawback that it is extremely sensitive to the way the belief-set is broken down into internally consistent subsets.[10] But our account is simpler, for we do not have to split up the agent. Even explicit inconsistencies are harmless on our account, since for all $m > 0$, $\Box_m(A \mathbin{\&} {\sim}A) \supset \Box_m B$ is invalid (in all $\mathscr{C}_nK$, $0 \leq n \leq \omega$).

Notice that our approach does *not* involve abandoning classical logic. We may retain $\mathscr{C}_0$ as a valid description of the actual world, but we must resist the temptation to regard the belief-set of an agent as necessarily being a theory of the same logic. (See §2.2 above for the meaning of "theory of a logic".) The belief-set need not be classical; the agent may adhere to another logic than we (the belief-ascribers) do. Just as the ascription of beliefs is, according to Clark (1976),

> mainly a matter of keeping the references and concepts of those of us who are scribes, recording the occurrences of psychical happenings, distinct from those of the agents to whom we ascribe mental events,—

so the ascription of beliefs is a matter of keeping the agents' and our (the scribes') *logics* distinct as well. We should not be so narrow-minded (or conceited) as to foist our own logic on everyone.

Are agents having different logics than ours *ipso facto* irrational? We do not think so. Rationality is as logic-relative as necessity. Whether a particular system of beliefs is rational or irrational just depends on the logic by which this system is judged, just as a sentence may be necessary according to one logic and contingent according to another. (For example, $\approx(A \mathbin{\&} {\sim}^{(n)}A)$ is necessary according to $\mathscr{C}_nK$, but contingent according to $\mathscr{C}_mK$ if $m > n$.) Let us say that a belief-set is *rational* iff it is a theory of *some* logic; it is *rational according to logic* $\mathscr{C}_nK$ iff it is a *theory of* $\mathscr{C}_nK$. So a belief-set containing $A \mathbin{\&} {\sim}^{(n)}A$, for example, cannot be rational according to $\mathscr{C}_nK$ while it may be rational according to $\mathscr{C}_mK$, $m > n$. (Whether it actually is rational according to $\mathscr{C}_mK$ depends, of course, not only on this sentence itself, but also on the rest of the belief-set.)

Thus, if our systems are given a doxastic interpretation, they represent various types of rational belief. For each logic $\mathscr{C}_n$ of the da Costa series there is a corresponding type of believer, whose beliefs are rational with respect to just that

[8] On classical doxastic logic see, e.g., Hintikka (1962), Lenzen (1978) and Lenzen (1980).

[9] See, e.g., Lewis (1982), Lewis (1986); postscript to Lewis (1978) in Lewis (1983), Rescher & Brandom (1980), Stalnaker (1984). The minimal deontic logic D of Chellas (1980) is a good example of a (deontic variant of a) doxastic logic that may be obtained in this way.

[10] This criticism has been expressed by Belnap, Jr. (1977).

logic. The belief-sets of these various types of believers are semantically modelled by different types of worlds. For each type of believer there is a different class of "doxastic alternatives" (as they are commonly called), worlds the believer "mentally lives in"; these worlds may be different from the type of worlds we imagine ourselves to be living in and they may accordingly be merely "imaginary" to us.

The range of forms of rationality we admit is, of course, rather limited: we have not included intuitionists, followers of Łukasiewicz's three-valued logic, etc. But our approach is at least not as parochial as that of the classical doxastic logicians, who see classical rationality as the only form of rationality, by which everyone is to be judged, even if the objects of the judgment themselves explicitly disavow the standards by which the judgment is made (as the intuitionists do).[11]

3.3. **Logical relativism.** Now, this recognition of a plurality of types of rational belief brings us close to the thesis of "logical relativism", which has received a tremendous amount of discussion within anthropology during the last 75 years. And indeed, we think our account manages to throw some long-needed light on this notoriously unclear thesis.

Logical relativists typically make the following claims.

1. "People of different cultures may have specifically different logics (for example, [there may be] a peculiarly Chinese logic distinct from Western logics)" (Lévy-Bruhl 1949). People of different cultures who follow different logics than ours should not be considered irrational: their "beliefs are on our standards irrational, but on other [. . .] standards they are about 'real' phenomena and 'logical"' (Lukes 1967). "The standards of rationality in different societies do not always coincide" (Winch 1964).
2. In an "ontological" formulation, logical relativism is the claim "that people of other cultures live in other worlds, so that what is rational in their world may well appear irrational in ours" (Sperber 1982). Sperber elaborates: "The relativist slogan, that people of different cultures live in different worlds, would be nonsense if understood as literally referring to physical worlds. If understood as referring to cognized worlds, it would overstate a very trivial point. [. . .] If, however, the worlds referred to are *cognizable worlds*, then the claim need be neither empty nor absurd." (Ibid.)
3. To these claims, it was, originally, often added that "the primitive mind is not constrained above else, as ours is, to avoid contradictions. What to our eyes is impossible or absurd, it sometimes will admit without seeing any difficulty." (Lévy-Bruhl 1925). "It does not bind itself down, as our thought does, to avoiding contradiction" (Lévy-Bruhl 1910).

It is of course an empirical matter to decide whether the thesis of logical relativism is true. Current opinion no longer seems to favor it.[12] However, this may at least partially be due to its unclarity: the thesis of logical relativism hinges on such

[11] The recognition of a variety of types of rational belief makes our systems different from the non-classical doxastic logics to be found in Routley & Routley (1975) and da Costa & French (n.d.). Similarly, it makes them different from modern "situation semantics" and "discourse representation theory", which are nowadays often put forward as successors to the modal, possible-worlds approach to doxastic logic. Our critique of single-operator "modal" doxastic logic applies with the same force to the latter approaches: even if we consider much weaker systems than classical logic—as these modern analyses do,—we should distinguish between the scribes' logics and the logics of the agents to whom the beliefs are ascribed. (One should distinguish between situations scribes and agents think they live in, or between discourses of scribes and agents, respectively.)

[12] As the textbook by Cole & Scribner (1974) states, "The most firmly based [. . .] conclusion we can reach [. . .] is that [. . .] there is no evidence for different kinds of reasoning processes such as the old classic theories alleged—we have no evidence for a 'primitive logic'." By the way, the thesis was also repudiated by its originator towards the end of his life (Lévy-Bruhl 1949).

notions as "logic", "rationality", "(cognizable) world", "consistency" and "contradiction", but anthropologists have always ignored the clarification of these notions in logic, while logicians showed no interest in clarifying the anthropological debates either. Therefore the thesis may have been abandoned too early. The merits and defects of a hypothesis cannot be properly judged until the hypothesis is sufficiently understood.

We think our "doxastic Vasil'evean" systems precisely enable us to clarify the three claims of logical relativism. First, we have seen how the claim that different people have "different logics" and "different standards of rationality" may be understood: their belief-sets are theories of different logics. Second, we have seen that theories of different logics describe different types of worlds. People having different logics do not have the same "doxastic alternatives" and may therefore be said to "live in" different kinds of worlds (mentally). Sperber's "cognizable worlds" are just the same as our "imaginable worlds". And finally, we have seen that some of the belief-sets we have considered (viz., the theories of the systems $\mathscr{C}_n K$, $n > 0$) are tolerant of contradictions, which provides a formal underpinning of the third claim. Therefore we think our analysis goes a long way in providing a clear and adequate explanatory model of the central traits of logical relativism.

4. Conclusion

This completes our exposition of multiply modal logics based on da Costa's $\mathscr{C}_n$, $0 \leq n \leq \omega$. We have not indicated all areas to which our systems might be applied. For example, the analysis of "truth in fiction" bears a close resemblance to doxastic logic (Lewis 1978), and our approach may be used to give an account of truth in fictional or non-fictional texts which do not subscribe to the canons of classical logic, but follow, describe or proclaim different logics. Think, for example, of tales written in accordance with paraconsistent logic, or simply of intuitionistic textbooks: it would be unfair, it would not be in accordance with the spirit of the texts, and it may even be seen as a sign of misunderstanding them, to judge such texts by classical logic. Deontic logic (which is also close to modal logic) would be another area of application. Various cultures might not only be pluralistic in their ethical norms (e.g., in the way described by Menger (1974)), but also in the logical standards by which they judge adherence to these norms.

Without doubt, there are more applications to be found. However, we hope the above may suffice to demonstrate the usefulness of the pluralistic, relativistic approach to modal logic. As Lewis (1986) has stated, the realm of possible worlds is "a philosophers' paradise", but he went on to argue that we do not need impossible worlds to carry out any interesting philosophical tasks. We hope to have shown that impossible worlds are as useful as possible worlds, and, moreover, that we do not need just one type, but lots and lots of varieties of them.[13]

Note added in print (to the original article)

As has already been pointed out in the text, the restriction to the da Costa series is inessential: our account may be extended to other systems of logic. Indeed, we have also constructed a system consisting of (1)the formalization of Vasil'ev's imaginary logic by Arruda (1977), (2) intuitionistic logic and (3) Łukasiewicz's three-valued logic.[14]

Professor N. C. A. da Costa has indicated how the above construction may be made completely general in one fell swoop. Loparić and he have demonstrated that

[13]The author wishes to thank professors N. C. A. da Costa and K. Sadegh-zadeh for their stimulating comments.

[14]Lokhorst (1985).

any system of logic whatsoever has a two-valued semantics of valuations relative to which it is sound and complete.[15] This has the consequence that all logical systems may be treated in exactly the same way as the da Costa systems have been treated here.

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