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Technical Report

# Deontic Linear Logic with Petri Net Semantics

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# Deontic Linear Logic with Petri Net Semantics*

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**Abstract** Deontic logic is the logic of obligation, permission and prohibition. Linear logic is a resource conscious logic which is well-known within computer science. Petri nets are models of concurrent dynamic processes which have been used in hundreds of applications. In this paper, we present a deontic linear system with Petri net semantics. This system has two advantages over modal and relevantist deontic logic: (1) it is free from many of the so-called 'paradoxes' which plague the latter approaches; (2) its semantics are related to modelling techniques which are actually used in practice.

**Keywords** Deontic logic, Eubouliatic logic, Linear logic, Petri nets, Relevance logic

## 1 Introduction

In 1996, Mark Brown presented a paper in which he described the 'paradoxes of cumulative obligations and permissions', as he called them [9]. These paradoxes rest on the observation that being obliged to do $A$ once is intuitively

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different from being obliged to do $A$ twice; similarly, having two obligations to do $A$ is intuitively different from having just one obligation to do $A$. The same applies, *mutatis mutandis*, to permission. Standard deontic logic cannot express these differences. One might try to represent being obliged to do $A$ twice by $O(A \wedge A)$ and having two obligations to do $A$ by $OA \wedge OA$, where $O$ stands for 'it is obligatory that'. But this won't work, since according to standard deontic logic the latter formulas are provably equivalent with $OA$.

After giving his lecture, Mark Brown was approached by several people who suggested to him that these paradoxes might conceivably be resolved by using *linear logic*. Linear logic is a 'resource conscious' logic which is well-known in computer science but less well known in philosophy [11, 17, 29]. It has a 'multiplicative' conjunction operator $\otimes$ such that $A$ is in general not equivalent with $A \otimes A$. Brown was clearly interested, but nobody could give him any reference because deontic linear logic is a subject which has not yet been studied.

In this paper, we will do two things. First, we will present a system of deontic linear logic and compare it with standard modal deontic logic and deontic relevance logic. It will turn out that the new system is free from several of the paradoxes (including Brown's cumulative paradoxes) which plague the latter systems.

Secondly, we will present a variant of the linear deontic system which is sound and complete with respect to a certain class of models defined in terms of *Petri nets*. Petri nets are models of concurrent, asynchronous, distributed, nondeterministic dynamic processes. They are not well-known in philosophy, but they have been used in hundreds of applications in the real world [24, 25, 26, 27]. It might therefore be argued that the semantics of our linear system are far less 'academic' than the algebraic, geometric and possible world semantics of modal and relevantist deontic logic.

# 2 Four Alethic Systems

The deontic systems we will discuss are based on the following alethic systems. Our notation is more or less the same as that of [7], but we write $\otimes$ instead of $\circ$.

1. System **LL**. Linear logic without exponentials [7].

    - Syntax. $A ::= \mathsf{1}|\mathsf{T}|\mathsf{F}|a|\neg A|A \otimes A|A \rightarrow A|A \wedge A|A \vee A$, where $a$ ranges over a countable set of atomic assertions AT. $\mathsf{1}$ is pronounced as 'one', $\mathsf{T}$ as 'true', $\mathsf{F}$ as 'false', $\neg$ as 'not', $\otimes$ as 'times',

$\to$ as 'entails', $\wedge$ as 'with' and $\vee$ as 'plus'. $\otimes$ and $\to$ are known as 'multiplicative' connectives, $\wedge$ and $\vee$ as 'additive' connectives.

- Definition.

  (a) $A \leftrightarrow B = (A \to B) \wedge (B \to A)$

- Axiomatization.

  (a) $(A \to B) \to ((B \to C) \to (A \to C))$
  (b) $(A \to (B \to C)) \to (B \to (A \to C))$
  (c) $(A \wedge B) \to A$
  (d) $(A \wedge B) \to B$
  (e) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
  (f) $A \to (A \vee B)$
  (g) $A \to (B \vee A)$
  (h) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
  (i) $A \to (B \to (A \otimes B))$
  (j) $(A \to (B \to C)) \to ((A \otimes B) \to C)$
  (k) $\neg\neg A \to A$
  (l) $(A \to \neg B) \to (B \to \neg A)$
  (m) $\mathsf{1}$
  (n) $\mathsf{1} \to (A \to A)$
  (o) $A \to \mathsf{T}$
  (p) $\mathsf{F} \to A$
  (q) $A, A \to B \ / \ B$
  (r) $A, B \ / \ A \wedge B$

2. Relevant system **R** [4, 5, 7].

- Syntax. Same as that of **LL**. $\otimes$ is pronounced as 'is co-tenable with', $\to$ as 'entails', $\wedge$ as 'and', and $\vee$ as 'or'. $\otimes$ and $\to$ are 'intensional', $\wedge$ and $\vee$ 'extensional'.

- Axiomatization. Same as that of **LL**, but add the following axioms:

  (a) $(A \to (A \to B)) \to (A \to B)$
  (b) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$

  The former axiom may be replaced by

  (a′) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

3. Modal system **S4** with strict implication as the only primitive intensional connective [12].

   - Syntax. $A ::= \mathsf{T}|\mathsf{F}|a|\neg A|A \to A|A \wedge A|A \vee A$. $\to$ is read as 'strictly implies'. The other symbols are pronounced as in **R**.
   - Definition.

     (a) $A \leftrightarrow B = (A \to B) \wedge (B \to A)$
   - Axiomatization.

     (a) $A \to A$
     (b) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
     (c) $(A \to B) \to (C \to (A \to B))$
     (d) $(A \wedge B) \to A$
     (e) $(A \wedge B) \to B$
     (f) $(A \to B) \to ((A \to C) \to (A \to (B \wedge C)))$
     (g) $A \to (A \vee B)$
     (h) $A \to (B \vee A)$
     (i) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
     (j) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$
     (k) $A \to \neg\neg A$
     (l) $(\neg A \to \neg B) \to (B \to A)$
     (m) $(A \to B) \to ((A \to \neg B) \to (A \to C))$
     (n) $A \to \mathsf{T}$
     (o) $\mathsf{F} \to A$
     (p) $A, A \to B \mathbin{/} B$

4. Propositional calculus **PC** [12].

   - Syntax. Same as that of **S4**. $\to$ is read as 'materially implies'. The other symbols are pronounced as in **S4**.
   - Axiomatization. Same as that of **S4**, but add the following axioms:

     (a) $((A \to B) \to A) \to A$
     (b) $A \to (B \to A)$

# 3 Four Deontic Systems

In order to turn the just-presented systems into deontic systems, we use a trick popularized (but not invented) by Alan Ross Anderson: we enrich the language with a constant assertion $\delta$ (read as 'the desideratum' or 'the good thing') and define the concepts of obligation $O$ and strong permission $S$ in terms of that assertion.

The following table presents the appropriate definitions and their first occurrences in the literature.

| Alethic System | Deontic System | Strong Permission | | Obligation | |
|---|---|---|---|---|---|
| **PC** | **DPC** | $SA = A \to \delta$ | [we] | $OA = \delta \to A$ | [8] |
| **S4** | **DS4** | $SA = A \to \delta$ | [1, 30] | $OA = \delta \to A$ | [15, 1] |
| **R** | **DR** | $SA = A \to \delta$ | [3] | $OA = \delta \to A$ | [2] |
| **LL** | **DLL** | $SA = A \to \delta$ | [we] | $OA = \delta \to A$ | [we] |

$SA$ may also be read as '$A$ is prudent', '$A$ is safe', or '$A$ is without risk'. Anderson used the name 'eubouliatic logic' for the logic of this concept [3].

The operators of strong prohibition, strong obligation, weak prohibition and weak permission may be defined by $F_S A = \neg SA$, $O_S A = F_S \neg A$, $FA = O\neg A$ and $PA = \neg FA$.

# 4 Comparison of the Deontic Systems

The just-defined four deontic systems are compared in table 1 (p. 6). A '$+$' means that the alethic expansion of the deontic formula is a theorem of the indicated system, a '$-$' that it is not. A '$\oplus$' ['$\ominus$'] means that the expanded formula is ill-formed but would turn into a theorem [non-theorem] if $\otimes$ were replaced by $\wedge$. Theoremhood may be established by giving a suitable derivation, which is easy in every case. Non-theoremhood in **LL** and **R** may be established by using the algebraic semantics presented in [7]; non-theoremhood in **S4** and **PC** may be established by using possible worlds semantics [14].

Table 1 makes it clear that **DLL** gives rise to a smaller number of paradoxes than the other systems do. Rows 9–12 and 21–24 show that **DLL** does not suffer from any of the Paradoxes of Cumulative Permissions and Obligations [9]. Rows 13 and 25 show that it avoids some of the Paradoxes of Conditional Permission and Obligation [6]. Rows 14 and 16, on the other hand, show that **DLL** is *not* free from the Good Samaritan Paradox and Ross's Paradox [6].

**DLL** does not seem to give rise to some fatal new paradox. McArthur [22] objected against theorem 17, but we simply regard it as convenient. Theorem 18 is particularly attractive. It is an axiom of eight systems on the Top Ten of modal deontic logic [6], whereas we get it for free.

| | Deontic Formula | Alethic Expansion | LL | R | S4 | PC |
|---|---|---|---|---|---|---|
| 1 | $(A \to B) \to (SB \to SA)$ | $(A{\to}B){\to}((B{\to}\delta){\to}(A{\to}\delta))$ | $+$ | $+$ | $+$ | $+$ |
| 2 | $(SA \wedge SB) \leftrightarrow S(A \vee B)$ | $((A{\to}\delta){\wedge}(B{\to}\delta)){\leftrightarrow}((A{\vee}B){\to}\delta)$ | $+$ | $+$ | $+$ | $+$ |
| 3 | $(SA \vee SB) \to S(A \wedge B)$ | $((A{\to}\delta){\vee}(B{\to}\delta)){\to}((A{\wedge}B){\to}\delta)$ | $+$ | $+$ | $+$ | $+$ |
| 4 | $(A \to SB) \to (B \to SA)$ | $(A{\to}(B{\to}\delta)){\to}(B{\to}(A{\to}\delta))$ | $+$ | $+$ | $-$ | $+$ |
| 5 | $(A \to SB) \leftrightarrow S(A \otimes B)$ | $(A{\to}(B{\to}\delta)){\leftrightarrow}((A{\otimes}B){\to}\delta)$ | $+$ | $+$ | $\ominus$ | $\oplus$ |
| 6 | $A \to SSA$ | $A{\to}((A{\to}\delta){\to}\delta)$ | $+$ | $+$ | $-$ | $+$ |
| 7 | $(A \to SA) \to SA$ | $(A{\to}(A{\to}\delta)){\to}(A{\to}\delta)$ | $-$ | $+$ | $-$ | $+$ |
| 8 | $(A{\to}SB){\to}((A{\to}B){\to}SA)$ | $(A{\to}(B{\to}\delta)){\to}((A{\to}B){\to}(A{\to}\delta))$ | $-$ | $+$ | $+$ | $+$ |
| 9 | $SA \to (SA \otimes SA)$ | $(A{\to}\delta){\to}((A{\to}\delta){\otimes}(A{\to}\delta))$ | $-$ | $+$ | $\oplus$ | $\oplus$ |
| 10 | $S(A \otimes A) \to SA$ | $((A{\otimes}A){\to}\delta){\to}(A{\to}\delta)$ | $-$ | $+$ | $\oplus$ | $\oplus$ |
| 11 | $(SA \otimes SA) \to SA$ | $((A{\to}\delta){\otimes}(A{\to}\delta)){\to}(A{\to}\delta)$ | $-$ | $-$ | $\oplus$ | $\oplus$ |
| 12 | $SA \to S(A \otimes A)$ | $(A{\to}\delta){\to}((A{\otimes}A){\to}\delta)$ | $-$ | $-$ | $\oplus$ | $\oplus$ |
| 13 | $SA \to (B \to SA)$ | $(A{\to}\delta){\to}(B{\to}(A{\to}\delta))$ | $-$ | $-$ | $+$ | $+$ |
| 14 | $(A \to B) \to (OA \to OB)$ | $(A{\to}B){\to}((\delta{\to}A){\to}(\delta{\to}B))$ | $+$ | $+$ | $+$ | $+$ |
| 15 | $(OA \wedge OB) \leftrightarrow O(A \wedge B)$ | $((\delta{\to}A){\wedge}(\delta{\to}B)){\leftrightarrow}(\delta{\to}(A{\wedge}B))$ | $+$ | $+$ | $+$ | $+$ |
| 16 | $(OA \vee OB) \to O(A \vee B)$ | $((\delta{\to}A){\vee}(\delta{\to}B)){\to}(\delta{\to}(A{\vee}B))$ | $+$ | $+$ | $+$ | $+$ |
| 17 | $(A \to OB) \leftrightarrow O(A \to B)$ | $(A{\to}(\delta{\to}B)){\leftrightarrow}(\delta{\to}(A{\to}B))$ | $+$ | $+$ | $-$ | $+$ |
| 18 | $O(OA \to A)$ | $(\delta{\to}((\delta{\to}A){\to}A))$ | $+$ | $+$ | $-$ | $+$ |
| 19 | $OOA \to OA$ | $(\delta{\to}(\delta{\to}A)){\to}(\delta{\to}A)$ | $-$ | $+$ | $-$ | $+$ |
| 20 | $O(A \to B) \to (OA \to OB)$ | $(\delta{\to}(A{\to}B)){\to}((\delta{\to}A){\to}(\delta{\to}B))$ | $-$ | $+$ | $+$ | $+$ |
| 21 | $OA \to (OA \otimes OA)$ | $(\delta{\to}A){\to}((\delta{\to}A){\otimes}(\delta{\to}A))$ | $-$ | $+$ | $\oplus$ | $\oplus$ |
| 22 | $OA \to O(A \otimes A)$ | $(\delta{\to}A){\to}(\delta{\to}(A{\otimes}A))$ | $-$ | $+$ | $\oplus$ | $\oplus$ |
| 23 | $(OA \otimes OA) \to OA$ | $((\delta{\to}A){\otimes}(\delta{\to}A)){\to}(\delta{\to}A)$ | $-$ | $-$ | $\oplus$ | $\oplus$ |
| 24 | $O(A \otimes A) \to OA$ | $(\delta{\to}(A{\otimes}A)){\to}(\delta{\to}A)$ | $-$ | $-$ | $\oplus$ | $\oplus$ |
| 25 | $OA \to (B \to OA)$ | $(\delta{\to}A){\to}(B{\to}(\delta{\to}A))$ | $-$ | $-$ | $+$ | $+$ |
| 26 | $OA \to OOA$ | $(\delta{\to}A){\to}(\delta{\to}(\delta{\to}A))$ | $-$ | $-$ | $+$ | $+$ |
| 27 | $(A \to B) \to O(A \to B)$ | $(A{\to}B){\to}(\delta{\to}(A{\to}B))$ | $-$ | $-$ | $+$ | $+$ |
| 28 | $O(A \to A)$ | $(\delta{\to}(A{\to}A))$ | $-$ | $-$ | $+$ | $+$ |

Table 1: Some theorems and non-theorems of **DLL**, **DR**, **DS4** and **DPC**

McArthur has called formula 20 a "*sine qua non* of any reasonable deontic system" [22, p. 153]. (He did not motivate this claim.) **DLL** does not have it as a theorem, but we do not think that **DLL** is unreasonable. We therefore think that McArthur's claim about 20 is wrong.

# 5 Petri Nets

One of the most interesting features of linear deontic logic is that it may be interpreted in terms of Petri nets. This gives it a connection with actual modelling practices which is lacking in the case of relevant and modal deontic logic.

Petri nets are models of dynamic processes in terms of types of resources, represented by *places* which can hold to arbitrary nonnegative multiplicity, and how these resources are consumed or produced by actions, represented by *transitions*. They are usually described in terms of multisets.

A *multiset* over a set $S$ is a function $M\colon S \mapsto \mathbb{N}$; $M$ is *finite* iff $\{s \in S\colon M(s) \neq 0\}$ is finite. In the rest of this paper, *multiset* will always mean *finite multiset*. $\mathscr{M}(S)$ denotes the set of finite multisets over $S$. $\underline{0}$, the *empty multiset*, is the multiset such that $\underline{0}(s) = 0$ for all $s \in S$. The *singleton multiset* of $s \in S$ is the multiset over $S$ with multiplicity 1 at $s$ and 0 elsewhere. Addition of multisets is defined by $(M + M')(s) = M(s) + M'(s)$ for all $s \in S$. The scalar multiplication $nM$, where $n \in \mathbb{N}$, is defined by $(nM)(s) = n \cdot M(s)$ for all $s \in S$.

A *Petri net* is a structure $N = \langle P, T, {}^\bullet(-), (-)^\bullet \rangle$, where $P$ and $T$ are sets such that $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$, and ${}^\bullet(-)$ and $(-)^\bullet$ are functions from $T$ to $\mathscr{M}(P)$. $P$ is the set of *places*, $T$ is the set of *transitions*, and ${}^\bullet t$ and $t^\bullet$ are the *pre* and *post multisets* of $t$. A multiset over $P$ is called a *marking*. Markings may change as transitions *fire*. Given markings $M$ and $M'$ and a multiset $U$ over $T$ (also called a *step*), we say that $U$ may fire $M'$ from $M$ iff there is some marking $M''$ such that $M = \sum_{t \in T} U(t)^\bullet t + M''$ and $M'' + \sum_{t \in T} U(t) t^\bullet = M'$. $M \; R \; M'$ means that there is a step which may fire $M'$ from $M$. The *reachability relation* $\Rightarrow$ is the reflexive transitive closure of $R$. The downwards closure of $M$ is defined by $\downarrow M = \{M' \in \mathscr{M}(P)\colon M' \Rightarrow M\}$. Similarly, $\downarrow H = \bigcup_{M \in H} \downarrow M$, where $H \subseteq \mathscr{M}(P)$. Given a Petri net $N$, $Q_N = \{\downarrow H\colon H \subseteq \mathscr{M}(P)\}$. A Petri net is *atomic* iff $M \Rightarrow \underline{0}$ implies $\underline{0} \Rightarrow M$ for any marking $M$. (A sufficient condition for atomicity is that every transition has a nonempty post multiset.)

# 6 Linear Logic with Petri Net Semantics

Engberg and Winskel have described a slightly non-standard variant of linear logic which can be completely interpreted in terms of atomic Petri nets [10]. Bringing their notation in line with ours, their system **LL[EW]** may be described as follows.

## 6.1 Syntax

The set of well-formed formulas WFF is defined by

$$A ::= \mathsf{1}|\mathsf{T}|\mathsf{F}|a|A \otimes A|A \to A|A \wedge A| \bigvee_{i \in I} A_i,$$

where $a$ ranges over a countable set of atomic assertions AT and $I$ over countable indexing sets drawn from $\omega$. $\bigvee$ is countable additive disjunction, with unit $\mathsf{F}$. Note that $\neg$ is absent.

## 6.2 Definitions

1. $\bigotimes\{A_1, \ldots, A_n\} = A_1 \otimes \ldots \otimes A_n$
2. $A^n = \overbrace{A \otimes \ldots \otimes A}^{n}$
3. $A_1 \vee A_2 = \bigvee_{i \in \{1,2\}} A_i$
4. $A \leftrightarrow B = (A \to B) \wedge (B \to A)$
5. $!\,A = A \wedge \mathsf{1}$
6. $\neg A = \,!\,A \to \mathsf{F}$
7. $\widehat{M} = \bigotimes_{M(a) \neq 0} a^{M(a)}$, where $M \in \mathscr{M}(\mathrm{AT})$

'!' is pronounced as 'of course'. This operator expresses the unlimited availability of a resource. An expression of the form $\widehat{M}$, where $M \in \mathscr{M}(\mathrm{AT})$, is called a *marking assertion*.

## 6.3 Semantics

A *Petri net valuation* for WFF is a structure $V = \langle N, \_ \rangle$, where $N$ is a countable atomic Petri net and the *naming function* _ is a mapping from AT onto the set of singleton multisets of $P$. Given $V$, the *denotation function* $[\ ]_V \colon \mathrm{WFF} \mapsto Q_N$ is defined as follows:

1. $[\mathsf{1}]_V = \downarrow\underline{0}$
2. $[\mathsf{T}]_V = \mathscr{M}(P)$
3. $[\mathsf{F}]_V = \emptyset$
4. $[a]_V = \downarrow\underline{a}$
5. $[A \otimes B]_V = \{M \in \mathscr{M}(P) \colon \exists M' \in [A]_V, M'' \in [B]_V.\, M \Rightarrow M' + M''\}$
6. $[A \to B]_V = \{M \in \mathscr{M}(P) \colon \forall M' \in [A]_V.\, M + M' \in [B]_V\}$
7. $[A \wedge B]_V = [A]_V \cap [B]_V$
8. $[\bigvee_{i \in I} A_i]_V = \bigcup_{i \in I} [A_i]_V$

$[A]_V$ may be regarded as the set of conditions (markings) which may establish $A$. Thus, $M$ may establish $A \otimes B$ iff $M$ may establish some condition which may establish $A$ plus some condition which may establish $B$. $M$ may establish $A \to B$ iff $M$ in addition to any condition which may establish $A$ may establish $B$. $M$ may establish $A \wedge B$ iff $M$ may establish both $A$ and $B$. $M$ may establish $\bigvee_{i \in I} A_i$ iff $M$ may establish some $A_i$, $i \in I$.

**Definitions**

1. $A \models_V B$ iff $[A]_V \subseteq [B]_V$
2. $\models_V A$ ($V$ *satisfies* $A$) iff $\mathsf{1} \models_V A$
3. $\models A$ ($A$ *is valid*) iff $\models_V A$ for all $V$
4. For $M \in \mathscr{M}(\mathrm{AT})$, $\underline{M}$ = the $M' \in \mathscr{M}(P)$ such that, for all $p \in P$, $M'(p) = \sum_{\underline{a}=p} M(a)$

**Observations**

1. $\models_V A$ iff $\underline{0} \in [A]_V$
2. $\models_V A \to B$ iff $[A]_V \subseteq [B]_V$
3. $[\neg A]_V = \begin{cases} \emptyset & \text{if } \models_V A \\ \mathscr{M}(P) & \text{otherwise} \end{cases}$
4. $\models_V \neg A$ iff $\not\models_V A$
5. $A \to \neg\neg A$ is valid, whereas $\neg\neg A \to A$ is not

**Structural rules**

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B} \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

**Logical rules**

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \frac{\Gamma \vdash A}{\Gamma, \mathsf{1} \vdash A} \quad \frac{}{\vdash \mathsf{1}}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\frac{\Gamma \vdash A_i \text{ for some } i \in I}{\Gamma \vdash \bigvee_{i \in I} A_i} \qquad \frac{\Gamma, B_i \vdash C \text{ for all } i \in I}{\Gamma, \bigvee_{i \in I} B_i \vdash C} \qquad \frac{}{\Gamma \vdash \mathsf{T}}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} \qquad \frac{}{\Gamma, \mathsf{F} \vdash A}$$

**Axioms specific to atomic nets**

$$\begin{array}{rcl}
(\bigvee_{i \in I} A_i) \wedge B & \vdash & \bigvee_{i \in I}(A_i \wedge B) \\
 & \vdash & A \vee \neg A \\
A & \vdash & \bigvee_{M \in \mathscr{M}(\mathrm{AT})}(\widehat{M} \otimes !(\widehat{M} \rightarrow A)) \\
\widehat{M} \rightarrow \bigvee_{i \in I} A_i & \vdash & \bigvee_{i \in I}(\widehat{M} \rightarrow A_i)
\end{array}$$

Table 2: System **LL[EW]**

6. $\underline{M} \Rightarrow \underline{M'}$ iff $\models_V \widehat{M} \rightarrow \widehat{M}'$, whereas $\underline{M} \not\Rightarrow \underline{M'}$ iff $\models_V \neg(\widehat{M} \rightarrow \widehat{M}')$

Observation 5 shows that **LL[EW]** is different from **LL**, in which $\neg\neg A \rightarrow A$ is an axiom. Observation 6 shows that both reachability and non-reachability are expressible within the language.

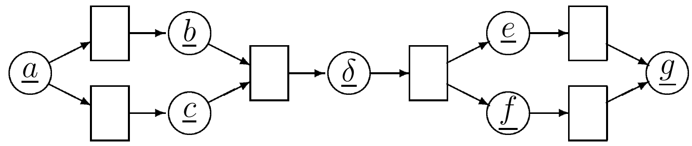
### 6.4 Proof System

Proof system **LL[EW]** is displayed in table 2 (p. 10). $\Gamma$ and $\Delta$ are multisets of assertions. It can be proven that $\vdash A$ iff $\models A$ [10].

## 7 Deontic Linear Logic with Petri Net Semantics

### 7.1 Basic Definition

Deontic system **DLL[EW]** is identical with **LL[EW]** except that the language contains a constant assertion $\delta$ and the definitions $SA = A \rightarrow \delta$ and $OA = \delta \rightarrow A$.




$[a]_V = \{\underline{a}\}$ $\quad$ $[b]_V = \{\underline{a}, \underline{b}\}$ $\quad$ $[c]_V = \{\underline{a}, \underline{c}\}$

$[\delta]_V = \{2\underline{a}, \underline{a}+\underline{b}, \underline{a}+\underline{c}, \underline{b}+\underline{c}, \underline{\delta}\}$ $\quad$ $[b \otimes c]_V = [\delta]_V \setminus \{\underline{\delta}\}$ $\quad$ $[b \wedge c]_V = \{\underline{a}\}$

$[b \vee c]_V = \{\underline{a}, \underline{b}, \underline{c}\}$ $\quad$ $[a \to \delta]_V = \{\underline{a}, \underline{b}, \underline{c}\}$ $\quad$ $[a^2]_V = \{2\underline{a}\}$

$[e]_V = \{\underline{e}\}$ $\quad$ $[f]_V = \{\underline{f}\}$ $\quad$ $[g]_V = \{\underline{e}, \underline{f}, \underline{g}\}$

$[e \otimes f]_V = [\delta]_V \cup \{\underline{e}+\underline{f}\}$ $\quad$ $[e \wedge f]_V = \emptyset$ $\quad$ $[e \vee f]_V = \{\underline{e}, \underline{f}\}$

$[\delta \to g]_V = \emptyset$ $\quad$ $[g^2]_V = [\delta]_V \cup \{p+q \colon p, q \in \{\underline{e}, \underline{f}, \underline{g}\}\}$

$\vDash_V \neg Sa$ $\quad$ $\vDash_V \neg Sb$ $\quad$ $\vDash_V \neg Sc$

$\vDash_V S\delta$ $\quad$ $\vDash_V S(b \otimes c)$ $\quad$ $\vDash_V \neg S(b \wedge c)$

$\vDash_V \neg S(b \vee c)$ $\quad$ $\vDash_V \neg SSa$ $\quad$ $\vDash_V S(a^2)$

$\vDash_V O\delta$ $\quad$ $\vDash_V \neg Oe$ $\quad$ $\vDash_V \neg Of$

$\vDash_V \neg Og$ $\quad$ $\vDash_V O(e \otimes f)$ $\quad$ $\vDash_V \neg O(e \wedge f)$

$\vDash_V \neg O(e \vee f)$ $\quad$ $\vDash_V \neg OOg$ $\quad$ $\vDash_V O(g^2)$

Figure 1: Sample deontic Petri net valuation

It may be noticed that $\vDash_V SA$ iff $[A]_V \subseteq [\delta]_V$ and that $\vDash_V OA$ iff $[\delta]_V \subseteq [A]_V$. Thus, $A$ is permitted iff any condition which may establish $A$ may establish $\delta$; $A$ is obligatory iff the converse holds.

It may also be noticed that if $A$ and $\delta$ are marking assertions, i.e., $A = \widehat{M_A}$ and $\delta = \widehat{M_\delta}$, then $\vDash_V SA$ iff $\underline{M_A} \Rightarrow \underline{M_\delta}$ and $\vDash_V OA$ iff $\underline{M_\delta} \Rightarrow \underline{M_A}$. Thus, $A$ is permitted iff the desirable marking is reachable from the marking named by $A$. $A$ is obligatory iff the converse holds. The former condition is intuitively attractive, the latter perhaps less so.

## 7.2 An Example

Figure 1 (p. 11) illustrates the semantics. Places are represented by circles, transitions by squares, and pre and post multisets by arcs of the appropriate multiplicities. It is assumed that $\mathrm{AT} = \{a, b, c, \delta, e, f, g\}$.

### 7.3 Other Deontic Notions

The concepts of strong prohibition, strong obligation, weak prohibition and weak permission are problematical within the present framework. The corresponding operators could be defined in the same way as before (section 3), but this would have some unfortunate consequences. The following formulas would, for example, be valid:

1. $F_S A \leftrightarrow (F_S A \otimes F_S A)$
2. $O_S A \leftrightarrow O_S(A \otimes A)$
3. $FA \leftrightarrow F(A \otimes A)$
4. $PA \leftrightarrow (PA \otimes PA)$

Thus, some paradoxes of deontic accumulation would reappear. Such desirable formulas as $\neg F_S A \rightarrow SA$ and $\neg PA \rightarrow FA$ would, on the other hand, be invalid. The classical negation operator of **DLL** does not lead to these difficulties, but it cannot easily be interpreted in terms of Petri nets. 'Cancellative linear logic' with its 'financial token games', in which there are not only resources but also *debts*, may offer a way out of this *impasse*, but this system has not yet been studied in sufficient detail [21].

### 7.4 Comparison of DLL[EW] with Other Deontic Systems

Table 1 still holds if **DLL** is replaced by **DLL[EW]**.

## 8 Conclusions

The linear logic *cum* Petri net approach towards deontic logic has several interesting features.

- It avoids some of the paradoxes—notably the paradoxes of cumulative permissions and obligations—which plague relevant and modal deontic logic.
- In contrast with the algebraic, geometric and possible worlds semantics of the latter systems—which are only of academic interest—the semantics of the last system we have presented are related to modelling techniques which have proven to be useful in practice.

- Because our approach is based on linear logic and Petri nets, it will look less unfamiliar to computer scientists than any of the systems which have thus far been recommended to them (see, e.g., [23]).

This is not to say that the particular system we have discussed is completely satisfactory. Certain problems were already noted in the previous section; it would, on the other hand, be interesting to add names for transitions, steps and sequences of steps to the language and to strive after some kind of integration with the results obtained by Linz [18]. Further work on deontic linear logic is therefore desirable.

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