Where Did Mally Go Wrong?

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Abstract. In 1926, Ernst Mally proposed the first system of deontic logic. His system turned out to be unacceptable. How can it be repaired? We discuss several proposals to reformulate it in terms of strict implication, relevant implication and strict relevant implication.

1 Introduction

In 1926, Ernst Mally proposed the first system of deontic logic [14]. He adopted the following principles:

1. If A requires B and if B then C, then A requires C.
2. If A requires B and if A requires C, then A requires B and C.
3. A requires B if and only if it is obligatory that if A then B.
4. The unconditionally obligatory is obligatory.
5. The unconditionally obligatory does not require its own negation.

Mally formalized these principles as follows. He used the language of the classical propositional calculus enriched with the connective O ("it is obligatory that") and the propositional constant $\boldsymbol{u}$ ("the unconditionally obligatory," *das unbedingt Geforderte*). He formalized "A requires B" as $A \supset OB$.

D1 $((A \supset OB) \,\&\, (B \supset C)) \supset (A \supset OC)$
D2 $((A \supset OB) \,\&\, (A \supset OC)) \supset (A \supset O(B \,\&\, C))$
D3a $(A \supset OB) \supset O(A \supset B)$
D3b $O(A \supset B) \supset (A \supset OB)$
D4 $O\boldsymbol{u}$
D5 $\sim(\boldsymbol{u} \supset O{\sim}\boldsymbol{u})$

Mally's deontic logic consisted of the classical propositional calculus plus axioms D1–D5. (Mally wrote D4 as $\exists \boldsymbol{u} O\boldsymbol{u}$, but this should be regarded as a mistake; see [6, pp. 2–3].)

Menger [15] showed that $A \equiv OA$ is a theorem of this system (Mally did not mention this formula), from which he concluded that it is unacceptable:

> This result seems to me to be detrimental for Mally's theory, however. It indicates that the introduction of the sign O is superfluous in the sense that it may be cancelled or inserted in any formula at any place

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> we please. But this result (in spite of Mally's philosophical justification) clearly contradicts not only our use of the word "ought" but also some of Mally's own correct remarks about this concept, e.g., the one at the beginning of his development to the effect that $A \supset (OB \vee OC)$ and $A \supset O(B \vee C)$ are not equivalent. Mally is quite right that these two propositions are not equivalent according to the ordinary use of the word "ought." But they are equivalent according to his theory by virtue of the equivalence of A and OA ([15, p. 58], notation adapted).

Menger's criticism is puzzling. The claim that "the sign O is superfluous in the sense that it may be cancelled or inserted in any formula at any place we please" is false because $A \vee B$ is well-formed but $AO \vee B$ is not. We conjecture that Menger intended to make the following objection:

Observation 1. *Mally's system is deontically trivial in the sense that it is closed under the following two rules:*

1. *from F to infer $F^{A}/_{OA}$,*
2. *from F to infer $F^{OA}/_{A}$,*

where $F^{A}/_{B}$ is the result of replacing one occurrence of A in F by B.

Mally's system is deontically trivial in this sense because it has the theorem $A \equiv OA$, as Menger showed, from which these two rules follow.

Where did Mally go wrong? How can Menger's and Mally's objections be avoided? We shall discuss several proposals. First, we shall show that reformulating it in terms of one of the Lewis systems of strict implication does not work: if all occurrences of material implication are replaced by strict implication, the resulting system is still objectionable for the reason pointed out by Menger. We do not exclude that one of the weaker Lemmon systems of strict implication might work (Section 2).

Second, we shall show that reformulating Mally's principles in terms of some system of relevant implication does lead to the desired result. This was already pointed out in [11] and [12]. We supplement this earlier proposal by the observation that choosing system **R** results in a deontic system that is weaker than necessary because it does not provide the relatively uncontroversial principle that obligation implies permission (where "it is permitted that A" is defined as "it is not obligatory that not A"). System **KR** of relevant logic (also known as "classical," "super-classical," "Boolean" or "super-Boolean" relevant logic) is better in this regard (Section 3).

Thirdly, we shall point out that the resulting system is unsatisfactory from a modern point of view. However, it can be reformulated in such a way that only one axiom needs modification to make it acceptable. This modification is similar to a modification suggested by Føllesdal and Hilpinen [6] (Section 4).

2 Strict Implication

We first study what happens if Mally's principles are reformulated in terms of strict implication. If one uses one of the Lewis systems, such as **S1°**, **S1**, **S2**, **S3**,

S4 or **S5** (see [20]), then such a reformulation does not work, as we shall now show.

System **S** is formulated in the language of the propositional calculus with an additional unary connective $\Box$, read as "necessarily," and the definitions $A \strictif B =_{\mathrm{df}} \Box(A \supset B)$ and $A \strictiff B =_{\mathrm{df}} (A \strictif B) \,\&\, (B \strictif A)$. $\strictif$ is read as "strictly implies." $\strictiff$ is read as "is strictly equivalent with."

Definition 1. ***S*** *has the following axioms and rules (named as in [4]):*

PL *The set of all tautologies.*
$\Box$PL $\{\Box A \colon A \in PL\}$.
US *Uniform substitution.*
RRSE$_{\mathbf{T}}$ *From* $A \strictiff B$ *and* F *to infer* $F^{A}/_{B}$.
MP *From* A *and* $A \supset B$ *to infer* B.
SMP *From* A *and* $A \strictif B$ *to infer* B.

Definition 2. *If A is a formula of Mally's deontic logic, then the* strict version *of A is the formula* $A^{\strictif}$ *got by replacing all occurrences of* $\supset$ *in* A *by* $\strictif$.

Observation 2. ***S*** *plus the strict versions of Mally's axioms D3a, D3b and D5 is deontically trivial in the sense of Observation 1.*

Proof [1]

1	$(A \,\&\, {\sim}A) \strictif O{\sim}(A \,\&\, {\sim}A)$	$\Box$PL
2	$O((A \,\&\, {\sim}A) \strictif {\sim}(A \,\&\, {\sim}A))$	1 D3a$^{\strictif}$ SMP
3	$((A \,\&\, {\sim}A) \supset {\sim}(A \,\&\, {\sim}A)) \strictiff (A \supset A)$	$\Box$PL PL
4	$O(A \strictif A)$	2 3 RRSE$_{\mathrm{T}}$
5	$A \strictif OA$	4 D3b$^{\strictif}$ SMP
6	$((A \,\&\, {\sim}B) \supset C) \strictiff ((A \,\&\, {\sim}C) \supset B)$	$\Box$PL PL
7	$(OA \,\&\, {\sim}B) \strictif OA$	$\Box$PL
8	$O((OA \,\&\, {\sim}B) \strictif A)$	7 D3a$^{\strictif}$ SMP
9	$O((OA \,\&\, {\sim}A) \strictif B)$	6 8 RRSE$_{\mathrm{T}}$
10	$(OA \,\&\, {\sim}A) \strictif OB$	9 D3b$^{\strictif}$ SMP
11	$(OA \,\&\, {\sim}OB) \strictif A)$	6 10 RRSE$_{\mathrm{T}}$
12	$OA \strictif OA$	$\Box$PL
13	$O(OA \strictif A)$	12 D3a$^{\strictif}$ SMP
14	${\sim}O(\boldsymbol{u} \strictif {\sim}\boldsymbol{u})$	D3b$^{\strictif}$ D5$^{\strictif}$ SMP
15	$O(OA \strictif A) \,\&\, {\sim}O(\boldsymbol{u} \strictif {\sim}\boldsymbol{u})$	13 14 PL
16	$OA \strictif A$	11 15 SMP
17	$A \strictiff OA$	5 16 PL
18	$OA \strictiff A$	16 5 PL
19	$F/F^{A}/_{OA}$	17 RRSE$_{\mathrm{T}}$
20	$F/F^{OA}/_{A}$	18 RRSE$_{\mathrm{T}}$

$\Box$

[1] This proof was inspired by the machine-generated derivation given in the Appendix.

S0.9° is axiomatized as **S** plus axiom $(A \dashv B) \dashv (\Box A \supset \Box B)$ (see [4]). **S** is contained in **S0.9°**, **S0.9**, **S1°**, **S1** and all the Lewis systems mentioned in [20], so the suggestion to reformulate Mally's assumptions in terms of one of these systems so as to avoid Menger's objection does not work.

On the other hand, some weaker system of strict implication might conceivably work, such as Lemmon's **S0.5** (which lacks $\mathrm{RRSE_T}$) or his **E**-systems, which have no theorem of the form $\Box A$ (see [8]). Since the proof of Observation 2 given above depends on $\mathrm{RRSE_T}$ and $\Box$PL, it does not go through in these weaker systems. So even though not *all* kinds of strict implication are suitable, *some* of them might be. The resulting systems would be so weak, however, that it is to be feared that almost all of the strict versions of theorems mentioned by Mally (cf. Observation 4 below) would share the fate of $A \dashv OA$ and $OA \dashv A$ and fail to be derivable.

3 Relevant Implication

Let us explore an alternative suggestion and study the consequences of formulating Mally's assumptions in terms of *relevant* rather than material or strict implication (see [11] and [12]). There is a great variety of systems of relevance logic. We confine our attention to **R** and **KR** (see [1] and [2]). $\rightarrow$ is the symbol for relevant implication. Relevant equivalence $\leftrightarrow$ is defined by $A \leftrightarrow B =_{\mathrm{df}} (A \rightarrow B) \,\&\, (B \rightarrow A)$.

Definition 3. *System* ***R*** *is axiomatized as follows:*

R1 $A \rightarrow A$
R2 $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
R3 $A \rightarrow ((A \rightarrow B) \rightarrow B)$
R4 $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
R5 $(A \,\&\, B) \rightarrow A$
R6 $(A \,\&\, B) \rightarrow B$
R7 $((A \rightarrow B) \,\&\, (A \rightarrow C)) \rightarrow (A \rightarrow (B \,\&\, C))$
R8 $A \rightarrow (A \vee B)$
R9 $B \rightarrow (A \vee B)$
R10 $((A \rightarrow C) \,\&\, (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$
R11 $(A \,\&\, (B \vee C)) \rightarrow ((A \,\&\, B) \vee C)$
R12 $(A \rightarrow {\sim}B) \rightarrow (B \rightarrow {\sim}A)$
R13 ${\sim}{\sim}A \rightarrow A$
R14 $A \leftrightarrow (\boldsymbol{t} \rightarrow A)$
R15 ${\sim}A \leftrightarrow (A \rightarrow \boldsymbol{f})$
R16 $A \rightarrow \boldsymbol{T}$
R17 $\boldsymbol{F} \rightarrow A$
→E *From* A *and* $A \rightarrow B$ *to infer* B
&I *From* A *and* B *to infer* $A \,\&\, B$.

$\boldsymbol{t}$ is the conjunction of all provable statements, $\boldsymbol{f}$ is its negation. $\boldsymbol{T}$ is the *Verum* (which Mally wrote as V), $\boldsymbol{F}$ is the *Falsum* (which Mally wrote as Λ).

Definition 4. *System* $\boldsymbol{KR}$ *is* $\boldsymbol{R}$ *plus* $(A \,\&\, {\sim}A) \to B$.

Definition 5. *If* A *is a formula of Mally's deontic logic, then the* relevant version *of* A *is the formula* $A^{\to}$ *got by replacing all occurrences of* $\supset$, V *and* Λ *by* $\to$, $\boldsymbol{T}$ *and* $\boldsymbol{F}$, *respectively.*

Definition 6. *Given relevant system* $\boldsymbol{X}$, $\boldsymbol{X}^{\delta}$ *is* $\boldsymbol{X}$ *plus the relevant versions of Mally's axioms D1–D5.*

Definition 7. *Given system* $\boldsymbol{X}^{\delta}$, $\boldsymbol{X}^{\delta+}$ *is* $\boldsymbol{X}^{\delta}$ *plus* $\boldsymbol{T} \to \boldsymbol{u}$.

Adding $\boldsymbol{T} \to \boldsymbol{u}$ leads to a better result in Observation 4 below.

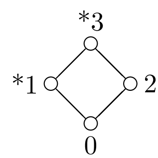
Observation 3. $\boldsymbol{R}^{\delta+}$ *and* $\boldsymbol{KR}^{\delta+}$ *do not provide the following formulas:*

1. $A \to OA$
2. $OA \to A$
3. $O(A \vee B) \to (OA \vee OB)$

Proof. By the computer program `MaGIC` [19], which generates the following tables and lattice, in which the designated values are preceded by a *.

	$\sim$	O
0	3	0
*1	2	0
2	1	1
*3	0	3

$\to$	0	1	2	3
0	3	3	3	3
1	0	1	2	3
2	0	0	1	3
3	0	0	0	3



& and $\vee$ are interpreted as lattice meet (greatest lower bound) and join (least upper bound), respectively. $\boldsymbol{F} = 0$, $\boldsymbol{t} = 1$, $\boldsymbol{f} = 2$, $\boldsymbol{u} = \boldsymbol{T} = 3$. All theorems of $\mathbf{KR}^{\delta+}$ receive the value 1 or 3, but $1 \to O1 = 0$, $O2 \to 2 = 2$ and $O(1 \vee 2) \to (O1 \vee O2) = 0$. □

$\mathbf{R}^{\delta+}$ and $\mathbf{KR}^{\delta+}$ therefore avoid both deontic triviality and the principle of distribution of obligation over disjunction, which Mally rejected (see the quote from Menger in Section 1 above, as well as [14, ch. 2, §4, p. 27, *ad* 3]).

Observation 4. *The relevant versions of 39 of the 47 theorems discussed by Mally (83%) are derivable in* $\boldsymbol{R}^{\delta+}$. *The relevant versions of the other 8 theorems (17%) are not derivable in* $\boldsymbol{R}^{\delta+}$. $\boldsymbol{KR}^{\delta+}$ *does not differ from* $\boldsymbol{R}^{\delta+}$ *in this respect.*

Proof. We follow the useful list given in [16, pp. 121–3]. A + indicates that the relevant version of Mally's theorem is a theorem of $\mathbf{R}^{\delta+}$, a − that it is not.

T1	$(A \to OB) \to (A \to O\boldsymbol{T})$	+
T2	$(A \to O\boldsymbol{F}) \leftrightarrow (\forall M)(A \to OM)$	+
T2′	$(A \to O\boldsymbol{F}) \to (\forall M)(A \to OM)$	+
T2″	$(\forall M)(A \to OM) \to (A \to O\boldsymbol{F})$	+

T3	$((A \to OB) \vee (A \to OC)) \to (A \to O(B \vee C))$	+
T4	$((A \to OB) \,\&\, (C \to OD)) \to ((A \,\&\, C) \to O(B \,\&\, D))$	+
T5a	$OA \leftrightarrow (\forall M)(M \to OA)$	−
T5b	$(\forall M)(M \to OA) \leftrightarrow (\forall M)(M \to OA)$	+
T6	$(OA \,\&\, (A \to B)) \to OB$	+
T7	$OA \to O\boldsymbol{T}$	+
T8	$((A \to OB) \,\&\, (B \to OC)) \to (A \to OC)$	+
T9	$(OA \,\&\, (A \to OB)) \to OB$	+
T10	$(OA \,\&\, OB) \leftrightarrow O(A \,\&\, B)$	+
T11	$((A \to OB) \,\&\, (B \to OA)) \leftrightarrow O(A \leftrightarrow B)$	+
T12a	$(A \to OB) \leftrightarrow (A \to OB)$	+
T12b	$(A \to OB) \leftrightarrow O(A \to B)$	+
T12c	$O(A \to B) \leftrightarrow O{\sim}(A \,\&\, {\sim}B)$	−
T12d	$O{\sim}(A \,\&\, {\sim}B) \leftrightarrow O({\sim}A \vee B)$	+
T13a	$(A \to OB) \leftrightarrow {\sim}(A \,\&\, {\sim}OB)$	−
T13b	${\sim}(A \,\&\, {\sim}OB) \leftrightarrow ({\sim}A \vee OB)$	+
T14	$(A \to OB) \leftrightarrow ({\sim}B \to O{\sim}A)$	+
T15	$(\forall M)(M \to O\boldsymbol{u})$	+
T16	$(\boldsymbol{u} \to A) \to OA$	+
T17	$(\boldsymbol{u} \to OA) \to OA$	+
T18	$OOA \to OA$	+
T19	$OA \leftrightarrow OOA$	−
T20	$(\boldsymbol{u} \to OA) \leftrightarrow ((A \to O\boldsymbol{u}) \,\&\, (\boldsymbol{u} \to OA))$	+
T21	$OA \leftrightarrow ((A \to O\boldsymbol{u}) \,\&\, (\boldsymbol{u} \to OA)))$	−
T22	$O\boldsymbol{T}$	+
T23	$(\boldsymbol{T} \to O\boldsymbol{u}) \,\&\, (\boldsymbol{u} \to O\boldsymbol{T})$	+
T23′	$\boldsymbol{T} \to O\boldsymbol{u}$	+
T23″	$\boldsymbol{u} \to O\boldsymbol{T}$	+
T23‴	$O(\boldsymbol{u} \leftrightarrow \boldsymbol{T})$	+
T24	$A \to OA$	−
T25	$(A \to B) \to (A \to OB)$	−
T26	$(A \leftrightarrow B) \to ((A \to OB) \,\&\, (B \to OA))$	−
T27	$(\forall M)({\sim}\boldsymbol{u} \to O{\sim}M)$	+
T27′	$(\forall M)({\sim}\boldsymbol{u} \to OM)$	+
T28	${\sim}\boldsymbol{u} \to O{\sim}\boldsymbol{u}$	+
T29	${\sim}\boldsymbol{u} \to O\boldsymbol{u}$	+
T30	${\sim}\boldsymbol{u} \to O\boldsymbol{F}$	+
T31	$({\sim}\boldsymbol{u} \to O\boldsymbol{F}) \,\&\, (\boldsymbol{F} \to O{\sim}\boldsymbol{u})$	+
T31′	$O({\sim}\boldsymbol{u} \leftrightarrow \boldsymbol{F})$	+
T32	${\sim}(\boldsymbol{u} \to O\boldsymbol{F})$	+
T33	${\sim}(\boldsymbol{u} \to \boldsymbol{F})$	+
T34	$\boldsymbol{u} \leftrightarrow \boldsymbol{T}$	+
T35	${\sim}\boldsymbol{u} \leftrightarrow \boldsymbol{F}$	+

The derivations of the theorems are easy. The refutations of the non-theorems are provided by the same matrices and lattice as in the proof of Observation 3. All theorems of $\mathbf{KR}^{\delta+}$ receive the value 1 or 3, but each of the following expressions gets the value 0:

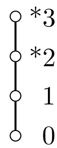
T5a	$O0 \leftrightarrow (0 \to O0)$	T21	$O2 \leftrightarrow ((2 \to O\boldsymbol{u}) \,\&\, (\boldsymbol{u} \to O2))$
T12c	$O(1 \to 0) \leftrightarrow O{\sim}(1 \,\&\, {\sim}0)$	T24	$1 \to O1$
T13a	$(1 \to O0) \leftrightarrow {\sim}(1 \,\&\, {\sim}O0)$	T25	$(1 \to 1) \to (1 \to O1)$
T19	$O2 \leftrightarrow OO2$	T26	$(1 \leftrightarrow 1) \to ((1 \to O1) \,\&\, (1 \to O1))$ □

Observation 5. $\boldsymbol{R}^{\delta+}$ *does not provide* $OA \to {\sim}O{\sim}A$. $\boldsymbol{KR}^{\delta+}$ *does.*

Proof. Proof of negative claim: by `MaGIC` [19], which generates the following tables and lattice:

	$\sim$	O
0	3	0
1	2	2
*2	1	2
*3	0	3

$\to$	0	1	2	3
0	3	3	3	3
1	0	2	2	3
2	0	1	2	3
3	0	0	0	3



& and $\vee$ are interpreted as in the proof of Observation 3. $\boldsymbol{F} = 0$, $\boldsymbol{f} = 1$, $\boldsymbol{t} = 2$, $\boldsymbol{u} = \boldsymbol{T} = 3$. All theorems of $\mathbf{R}^{\delta+}$ receive the value 2 or 3, but $O1 \to {\sim}O{\sim}1 = 1$. Proof of positive claim:

1	$OA \leftrightarrow ({\sim}O\boldsymbol{f} \to A)$	See [9] and [10]
2	$(A \to B) \to ((A \to {\sim}B) \to (A \to (B \,\&\, {\sim}B)))$	**R**
3	$(A \to (B \,\&\, {\sim}B)) \to (A \to (\boldsymbol{u} \to {\sim}\boldsymbol{u}))$	**KR**
4	$OA \to (O{\sim}A \to O(\boldsymbol{u} \to {\sim}\boldsymbol{u}))$	1 2 3
5	${\sim}O(\boldsymbol{u} \to {\sim}\boldsymbol{u})$	$\text{D3b}^{\to}$ $\text{D5}^{\to}$
6	$OA \to {\sim}O{\sim}A$	4 5 **R** □

Observation 5 is interesting because $OA \to {\sim}O{\sim}A$ is the relevant version of $OA \supset {\sim}O{\sim}A$. This theorem is often regarded as characteristic for deontic logic. There can be no doubt that Mally accepted it, even though he did not endorse it explicitly. His approval becomes clear from his discussion of the logic of willing (bouletic or boulomaic logic), which he saw as formally similar to deontic logic, but in which OA is read as "it is correctly desired that A." In this context, he gave the following paraphrase of D5: "somebody who correctly desires that A does not desire that not A; (not even implicitly); correct desire is free of contradiction" (*Wer richtig will, will nicht (auch nicht impliziterweise) das Negat des Gewollten; richtiges Wollen ist widerspruchsfrei*, [14, ch. 4, §10, p. 49, *ad* (V)]). This is a paraphrase of $OA \supset {\sim}O{\sim}A$ rather than D5.

4 Relevant Strict Implication

Observation 6. $\boldsymbol{KR}^{\delta}$ *has the following theorems:*

(a) $A \to O{\sim}O{\sim}A$
(b) $OOA \to A$

(c) $O(OA \vee B) \to (A \vee OB)$

Proof. (a) follows from D1$^{\to}$, D3a$^{\to}$ and D3b$^{\to}$: see [10]. (b) follows from $OA \to {\sim}O{\sim}A$ (Observation 5). (c) follows from theorem

KR2 $(A \mathbin{\&} ((B \circ C) \circ D)) \to (((A \circ C) \mathbin{\&} (B \circ D)) \circ C)$,

where $A \circ B =_{\text{df}} {\sim}(A \to {\sim}B)$, which is proven in [2, §54.1, pp. 263–4]. The details are as follows:

1	$(C \to ((C \to A) \vee (D \to B))) \to (A \vee (D \to (C \to B)))$	KR2 Df$\circ$ **R**
2	$({\sim}O\boldsymbol{f} \to (({\sim}O\boldsymbol{f} \to A) \vee (\boldsymbol{t} \to B))) \to (A \vee (\boldsymbol{t} \to ({\sim}O\boldsymbol{f} \to B)))$	1
3	$OA \leftrightarrow ({\sim}O\boldsymbol{f} \to A)$	[9] [10]
4	$O(OA \vee B) \to (A \vee OB)$	2 3 **R**

□

Because (a), (b) and (c) are unacceptable from a modern point of view (cf. [7]), $\mathbf{KR}^{\delta}$ is not acceptable either. This shortcoming can be removed as follows.

Definition 8. *$\boldsymbol{KR}^{O}$ consists of **KR** plus the following axioms:*

O1 $(A \to B) \to (OA \to OB)$
O2 $O(OA \to A)$
O3 $O\boldsymbol{u}$
O4 $OA \to {\sim}O{\sim}A$

Observation 7. *$\boldsymbol{KR}^{O}$ has the same theorems as $\boldsymbol{KR}^{\delta}$.*

Proof. O1 and O2 follow from D1$^{\to}$, D3a$^{\to}$ and D3b$^{\to}$: see [12]. O3 follows from D4$^{\to}$. O4 follows from Observation 5. D1$^{\to}$, D2$^{\to}$, D3a$^{\to}$ and D3b$^{\to}$ follow from O1 and O2: see [10] and [12]. D4$^{\to}$ follows from O3. D5$^{\to}$ follows from O1–O4 because:

1	$O\boldsymbol{u}$	O3
2	$\boldsymbol{u} \to {\sim}(\boldsymbol{u} \to {\sim}\boldsymbol{u})$	**R**
3	$O{\sim}(\boldsymbol{u} \to {\sim}\boldsymbol{u})$	1 2 O1
4	${\sim}O(\boldsymbol{u} \to {\sim}\boldsymbol{u})$	3 O4
5	${\sim}(\boldsymbol{u} \to O{\sim}\boldsymbol{u})$	4 D3a$^{\to}$

□

Definition 9. *Relevant alethic modal system $\boldsymbol{KR}^{\Box}$ has the following axioms and rules in addition to those of **KR**:*

□K $\Box(A \to B) \to (\Box A \to \Box B)$
□C $(\Box A \mathbin{\&} \Box B) \to \Box(A \mathbin{\&} B)$
□T $\Box A \to A$
□4 $\Box A \to \Box\Box A$
Nec *From A to infer $\Box A$.*

This system is the same as $\mathbf{R}^{\Box}$ [1, §27.1.3, pp. 343–4] plus $(A \mathbin{\&} {\sim}A) \to B$.

Definition 10. $A \Rightarrow B =_{df} \Box(A \rightarrow B)$

The symbol $\Rightarrow$ represents *relevant strict implication*, which is stronger than classical strict implication and relevant implication in the sense that both $(A \Rightarrow B) \rightarrow (A \prec B)$ and $(A \Rightarrow B) \rightarrow (A \rightarrow B)$ are theorems, where $A \prec B =_{\text{df}} \Box(A \supset B)$ and $A \supset B =_{\text{df}} {\sim}A \vee B$.

Definition 11. *Mixed alethic-deontic system* $\boldsymbol{KR}^{\Box O}$ *has the following axioms in addition to those of* $\boldsymbol{KR}^{\Box}$*:*

O1′ $(A \Rightarrow B) \rightarrow (OA \rightarrow OB)$
O2–O4 *As above (Definition 8)*

Observation 8. $\boldsymbol{KR}^{\Box O}$ *does not provide the formulas mentioned in Observations 3 and 6.*

Proof. By `MaGIC` [19]. The details are as follows.

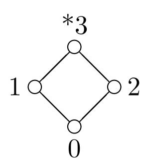
1. Refutation of the formulas mentioned in Observation 3: identical with the proof of Observation 3, but add:

A	0	*1	2	*3
$\Box A$	0	1	2	3

2. Refutation of the formulas mentioned in Observation 6: by the following tables and lattice, in which 3 is the only designated value:

	$\sim$	$\Box$	O
0	3	0	0
1	2	0	3
2	1	0	0
*3	0	3	3

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	2	3	2	3
2	1	1	3	3
3	0	1	2	3



$\boldsymbol{F} = \boldsymbol{f} = 0, \boldsymbol{t} = \boldsymbol{T} = \boldsymbol{u} = 3$. All theorems get the value 3, but $2 \rightarrow O{\sim}O{\sim}2 = 1$, $OO1 \rightarrow 1 = 1$ and $O(O1 \vee 2) \rightarrow (1 \vee O2) = 1$. $\Box$

Observation 8 implies that we have to make only one change in $\mathbf{KR}^{O}$ (replace O1 by O1′) to obtain a system that is not only acceptable in light of Menger's and Mally's criteria, but also from a modern point of view.

Føllesdal and Hilpinen have suggested that Mally's system can be repaired by replacing axiom D1 by the weaker D1′:

D1 $((A \supset OB) \,\&\, (B \supset C)) \supset (A \supset OC)$
D1′ $((A \supset OB) \,\&\, (B \prec C)) \supset (A \supset OC)$,

where $\prec$ is "some sort of strict implication," for example strict implication as defined in modal system **S4** (see [6, pp. 5–6]). This replacement of D1 by D1′ is similar to the replacement of O1 by O1′.

5 Historical Digression

Some of the observations we have made could already have been made in 1928, right after Ivan Orlov had made the first contribution to relevant logic (see [17]; on Orlov, see [3]). Orlov's system **OR** has the same theorems as the implication-negation fragment of **R** (see [5]). His relevant modal system $\mathbf{OR}^{\Box}$ (called **ORS4** in [5]) consists of **OR** plus axioms $\Box$K, $\Box$T, $\Box$4 and rule Nec (as in Definition 9). We define $\mathbf{OR}^{O}$ as **OR** plus axioms O1–O4 (as in Definition 8) and $\mathbf{OR}^{\Box O}$ as $\mathbf{OR}^{\Box}$ plus axioms O1′–O4 (as in Definition 11). Neither $\mathbf{OR}^{O}$ nor $\mathbf{OR}^{\Box O}$ provides $A \rightarrow OA$ or $OA \rightarrow A$ (by Observation 3). $\mathbf{OR}^{O}$ provides the dubious theorems $A \rightarrow O{\sim}O{\sim}A$ and $OOA \rightarrow A$ (proof: identical with the proof of Observation 6, but use O4 instead of Observation 5). $\mathbf{OR}^{\Box O}$ does not provide these theorems (by Observation 8).

6 Conclusion

Mally formulated the first system of deontic logic, but his system was unacceptable (Section 1). Where did he go wrong? It can be argued that it was his adoption of classical propositional logic that led him into trouble. Adopting some system of strict implication does not help as long as one uses one of the Lewis systems (Section 2). Adopting a system of relevance logic does; system **KR** is particularly suitable (Section 3). The resulting relevant deontic system expresses Mally's assumptions while it avoids both Menger's and Mally's own objections. It is not satisfactory from a modern point of view, but it can be reformulated in such a way that only one axiom needs modification to make it acceptable (Section 4).

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Appendix

Proof of $A \strictiff OA$ generated by `Prover9` [13]. The input is on lines 1–14, the output on lines 15–50. The methodology is explained in [18]. For readability, the formulas were converted from prefix notation to infix notation by a `Perl` script. As in [4], RRTE refers to the rule of replacement of tautological equivalents.

1	$\vdash (A \strictif OA) \,\&\, (OA \strictif A)$	goal
2	$\nvdash A$ or $\nvdash B$ or $\vdash A \,\&\, B$	MP PL
3	$\nvdash A$ or $\nvdash A \strictif B$ or $\vdash B$	SMP
4	$\vdash (A \,\&\, B) \strictif A$	$\Box$PL
5	$\vdash (A \,\&\, (A \supset B)) \strictif B$	$\Box$PL
6	$\nvdash F$ or $\vdash F^{\sim\sim A}/_{A}$	RRTE PL
7	$\nvdash F$ or $\vdash F^{(\sim A \supset \sim B)}/_{(B \supset A)}$	RRTE PL
8	$\nvdash F$ or $\vdash F^{((A\&B) \supset C)}/_{(A \supset (B \supset C))}$	RRTE PL

9	$\nvdash F$ or $\vdash F^{(\sim A \supset A)}/_{A}$	RRTE PL
10	$\nvdash F$ or $\vdash F^{\sim(A \& \sim B)}/_{(A \supset B)}$	RRTE PL
11	$\nvdash F$ or $\vdash F^{\Box(A \supset B)}/_{(A \strictif B)}$	Def $\strictif$
12	$\vdash (A \strictif OB) \strictif O(A \strictif B)$	D3a$^{\strictif}$
13	$\vdash O(A \strictif B) \strictif (A \strictif OB)$	D3b$^{\strictif}$
14	$\vdash \sim(\boldsymbol{u} \strictif O{\sim}\boldsymbol{u})$	D5$^{\strictif}$
15	$\nvdash (a \strictif Oa) \,\&\, (Oa \strictif a)$	deny 1
16	$\nvdash F$ or $\vdash F^{(A \supset \sim B)}/_{(B \supset \sim A)}$	para 6 7
17	$\nvdash F$ or $\vdash F^{(\sim A \supset B)}/_{(\sim B \supset A)}$	para 6 7
18	$\nvdash F$ or $\vdash F^{(A \supset \sim A)}/_{\sim A}$	para 6 9
19	$\nvdash F$ or $\vdash F^{\sim(A \supset B)}/_{(A \& \sim B)}$	para 10 6
20	$\nvdash F$ or $\vdash F^{\sim(A \& B)}/_{(A \supset \sim B)}$	para 6 10
21	$\nvdash F$ or $\vdash F^{(\sim A \strictif \sim B)}/_{(B \strictif A)}$	para 7 11, rewrite 11, flip
22	$\nvdash F$ or $\vdash F^{((A \& B) \strictif C)}/_{(A \strictif (B \supset C))}$	para 8 11, rewrite 11, flip
23	$\vdash O((OA \,\&\, B) \strictif A)$	hyper 3 4 12
24	$\nvdash F$ or $\vdash F^{(A \strictif \sim A)}/_{\Box \sim A}$	para 18 11, flip
25	$\vdash O\Box{\sim}A \strictif (A \strictif O{\sim}A)$	para 24 13
26	$\nvdash F$ or $\vdash F^{(\sim A \strictif B)}/_{(\sim B \strictif A)}$	para 17 11, rewrite 11
27	$\nvdash F$ or $\vdash F^{(A \& A)}/_{A}$	para 18 19, rewrite 6 6, flip
28	$\nvdash F$ or $\vdash F^{(A \& B)}/_{(B \& A)}$	para 16 19, rewrite 19 6 6
29	$\vdash O(OA \strictif A)$	para 27 23
30	$\vdash O((A \,\&\, OB) \strictif B)$	para 28 23
31	$\nvdash \sim A$ or $\nvdash B \strictif A$ or $\vdash \sim B$	para 21 3
32	$\vdash \sim A \strictif (A \supset \sim B)$	para 21 4, rewrite 20
33	$\vdash \sim A \strictif (A \supset B)$	para 6 32
34	$\vdash (A \,\&\, \sim A) \strictif B$	para 6 33, rewrite 22
35	$\vdash O((A \,\&\, \sim A) \strictif B)$	hyper 3 34 12
36	$\vdash O(A \strictif A)$	para 24 35, rewrite 20 6 11
37	$\vdash A \strictif OA$	hyper 3 36 13
38	$\nvdash Oa \strictif a$	ur 2 37 15
39	$\nvdash A \,\&\, (A \supset (Oa \strictif a))$	ur 3 5 38
40	$\vdash O(A \strictif (OB \supset B))$	para 22 30
41	$\vdash O((\sim A \,\&\, OA) \strictif B)$	para 26 40, rewrite 19 28
42	$\vdash (\sim A \,\&\, OA) \strictif OB$	hyper 3 41 13
43	$\vdash \sim OA \strictif (OB \supset B)$	para 21 42, rewrite 20 16 6
44	$\vdash (\sim OA \,\&\, OB) \strictif B$	para 22 43
45	$\vdash (OA \,\&\, \sim OB) \strictif A$	para 28 44
46	$\vdash OA \strictif (\sim OB \supset A)$	para 22 45
47	$\vdash \sim OA \supset (OB \strictif B)$	hyper 3 29 46
48	$\nvdash \sim OA$	ur 2 47 39
49	$\vdash \sim O\Box{\sim}\boldsymbol{u}$	hyper 31 14 25
50	$\bot$	resolve 49 48