ONTOLOGY, SEMANTICS, AND PHILOSOPHY OF MIND IN WITTGENSTEIN'S *TRACTATUS*: A FORMAL RECONSTRUCTION

GERT-JAN C. LOKHORST

Abstract. This paper presents a formal explication of Wittgenstein's early views on ontology, the syntax and semantics of an ideal logical language, and the propositional attitudes. It is shown that Wittgenstein gave a "language of thought" analysis of propositional attitude ascriptions, and that his ontological views imply that such ascriptions are truth-functions of (and supervenient upon) elementary sentences. Finally, an axiomatization of a quantified doxastic modal logic corresponding to Tractarian semantics is given.

First published in *Erkenntnis* **29** (1988), 35–75. ISSN 0165–0106.

0. Introduction

Historically, Wittgenstein's *Tractatus* is primarily a forerunner of Tarski's and Carnap's later contributions to semantics. However, the latter do not faithfully reflect Wittgenstein's ideas: for example, Wittgenstein's idea that predicates are names of properties is absent from Tarski's work, while Carnap's "state-descriptions" are certainly different from descriptions of states of affairs in the Tractarian sense (cf. Section 3 below). Therefore the question arises: is it possible to develop a semantical system which is both faithful to the *Tractatus* and as precise as Tarski's and Carnap's contributions? This is the question which we shall try to answer in the present paper. The effort will be rewarding: not only will it turn out that it is indeed *possible* to give a formal reconstruction, it will moreover appear that such a reconstruction has various features which are still interesting today. Thus, it not only yields a *truth-functional* analysis of quantification, modalities and propositional attitude ascriptions, it also shows that the *Tractatus* contains a quite modern language of thought theory and even a variant of the currently popular doctrine of psychophysical supervenience.

Our formal reconstruction of the *Tractatus* is not the first one to appear. As early as 1966, Stegmüller—condemning the average interpretation of the *Tractatus* as nothing but "a bunch of very unclear statements, which should first be explicated themselves" (Stegmüller 1966)—gave a formalization of the picture theory; shortly after, Suszko (1968), Wolniewicz (1968) and Mudersbach (1968) began to formalize Tractarian ontology. The formal approach has been taken up by perhaps a dozen philosophers since then. However, none of the previous contributions is wholly successful. First, none of them gives a comprehensive formalization of *both* object ontology *and* situation ontology *and* semantics; as a result, the *interrelations* between such subjects as the independence of states of affairs, the describability of the world by elementary sentences and the principle of truth-functionality have remained unclear. Secondly, Wittgenstein's remarks on propositional attitude ascriptions have never been discussed in formal terms before,[1] let alone his claim that they are truth-functional or the question whether they are definable in terms

Date: 1988.

[1]Exceptions are Lokhorst (1985 *a*), Lokhorst (1985 *b*), the precursors of the present paper.

of elementary sentences. Finally, all previous reconstructions are rather inelegant. The present reconstruction certainly avoids the first two defects; we hope it avoids the third too.

The plan of the paper is as follows. Because of the primacy of the ontological in the *Tractatus*, we start with this subject in Section 1. Section 2 discusses the syntax and Section 3 the semantics of sentences and pictures; the propositional attitudes are treated separately in Section 4. Section 5 presents the logical system the preceding results lead up to. Finally, the moral will be drawn in Section 6. Comparisons with earlier formalizations will continually be made as we go along.

1. The Ontology of the *Tractatus*

1.1. **Objects and States of Affairs.** For Wittgenstein, "objects" (*Gegenstände*) are the basic building-blocks of the world. They are the "substance of the world" (TLP[2] 2.021 ff.); all possible worlds have the same substance (TLP 2.022, 2.023, 2.024). The number of objects cannot be determined *a priori*; "it is a matter of physics to find out" (NB p. 127). However, Wittgenstein assumes the existence of at least one object (TLP 2.0211–2.0212, 2.026, 4.2211); on the other hand, he never refers to more than $\aleph_0$ objects (TLP 4.1272; NB p. 127). Denoting the set of objects (*Gegenstände*) by "**G**", we therefore stipulate:

Definition 1: $\mathbf{G}$ is a set such that $1 \leq Card(\mathbf{G}) \leq \aleph_0$.

It is important to realize that the category of "objects" is a very general one. Relations and properties, if there are such things, are objects too: "Auch Relationen und Eigenschaften etc. sind *Gegenstände*" (NB 16.6.15); "'Objects' also include relations; [. . .] 'thing' and relation are on the same level" (Lee 1980, p. 120). (This is Wittgenstein's so-called "realism" about relations and properties.) It cannot be settled *a priori* what kinds of objects there are; this can only be found out by empirical investigation, not by logic. Therefore we shall not explicitly distinguish between different kinds of objects and treat them all on a par.

The next step in the Tractarian composition of the world is constituted by "states of affairs" (*Sachverhalte*). States of affairs are concatenations of objects (TLP 2.03; cf. TLP 2.01, 2.0272, 3.21). Wittgenstein seems to have been uncertain as regards the maximum complexity a state of affairs may have. At first, he seems to have accepted only *finite* concatenations of objects: "The *infinitely* complex state of affairs seems to be a monstrosity!" (NB 23.5.15). Later on, however, he seems to have abandoned his repugnance to infinitely complex states of affairs (TLP 4.2211). We adopt the earlier view for the sake of simplicity.

In order to define the set of states of affairs **SA**, we first introduce the set $\mathbf{G*}$ of all finite concatenations of members of **G**.

Definition 2: $\mathbf{G*}$ is the smallest set such that:

(a) if $\mathbf{g}, \mathbf{g'} \in \mathbf{G}$, then $\mathbf{g} * \mathbf{g'} \in \mathbf{G*}$;

(b) if $\mathbf{g} \in \mathbf{G}$ and $\mathbf{s} \in \mathbf{G*}$, then $\mathbf{g} * \mathbf{s} \in \mathbf{G*}$.

Notice that $Card(\mathbf{G*}) = \aleph_0$, *even if* $Card(\boldsymbol{G}) = 1$! Therefore Suszko (1968, p. 24) (following Wolniewicz) made an error in claiming that "if there were finitely many objects then there would exist only finitely many configurations of them, i.e., finitely many states of affairs".

$\mathbf{G*}$ is, in general, *not* the set of states of affairs, as might be supposed. Certain additional restrictions may exclude some concatenations of objects from being states of affairs. For example, if **f** is a property and **g** a particular ($\mathbf{f}, \mathbf{g} \in \mathbf{G}$), then $\mathbf{f} * \mathbf{g}$ may well be a state of affairs, viz., the situation that **g** and **f** are concatenated, or

[2] Here and in the following, "TLP" stands for the *Tractatus* (Wittgenstein 1971). "NB" stands for the second edition of the *Notebooks*, including the appendices (Wittgenstein 1979). We shall occasionally provide our own translations.

the situation that $\mathbf{g}$ has property $\mathbf{f}$. But in this case, $\mathbf{g} * \mathbf{f}$ will presumably not be a situation at all. (It might be one if $\mathbf{g}$ were a second-order property.) The same goes for relations. If $\mathbf{R} \in \mathbf{G}$ is an n-ary relation, then $\mathbf{R} * \mathbf{g}_0 * \ldots * \mathbf{g}_n$ is the state of affairs that $\mathbf{R}$, $\mathbf{g}_0, \ldots, \mathbf{g}_n$ are concatenated, the situation that $\mathbf{R}$ is exemplified by $\mathbf{g}_0, \ldots, \mathbf{g}_n$, or, as Suszko (1968, p. 22) expresses it, the $\mathbf{R}$-configuration of $\mathbf{g}_0, \ldots, \mathbf{g}_n$. In this case, $\mathbf{g}_0 * \mathbf{R} * \mathbf{g}_1$ will presumably not be a situation at all.

However, we cannot give an *a priori* list of conditions an element of $\mathbf{G^*}$ must meet if it is to be counted as a "well-formed" state of affairs: we do not even know, for example, whether there are binary relations or not, for this is an empirical matter. Therefore we simply stipulate that $\mathbf{SA} \subseteq \mathbf{G^*}$.[3]

What is the cardinality of $\mathbf{SA}$? In the first place, $\mathbf{SA} \neq \emptyset$. This follows from the requirement in TLP 2.011 that each object occurs in some situation. Since $\mathbf{G} \neq \emptyset$, $\mathbf{SA}$ cannot be empty either.

However, we can do better than this by taking TLP 4.463 into account: here mention is made of "infinite logical space". As logical space is generated by $\mathbf{SA}$ (Section 1.3), $\mathbf{SA}$ must be infinite too. Therefore we stipulate:

Definition 3: $\mathbf{SA} \subseteq \mathbf{G^*}$ is a set such that:

(a) for each $\mathbf{g} \in \mathbf{G}$ there is at least one $\mathbf{s} \in \mathbf{SA}$ such that $\mathbf{s} = \mathbf{g}_0 * \ldots * \mathbf{g}_n$ and $\mathbf{g} = \mathbf{g}_i$ for some i, $0 \leq i \leq n$, and

(b) $\mathit{Card}(\mathbf{SA}) = \aleph_0$.

Clause (b) is important in connection with propositional attitude ascriptions: it implies that these are, in general, not definable in terms of elementary sentences (see note 27 below).

1.2. **Situations, Facts and Worlds.** The essential clue to understand Tractarian situation ontology has been provided by Suszko (1968): Tractarian situations are the elements of a complete atomic Boolean algebra. We shall adopt this suggestion, but turn Suszko's algebra "upside down" (i.e., consider its dual) as this leads to a more natural conception of possible worlds. The latter now become the "mereological sums" (suprema) of the possible situations they contain as "parts".[4] Given $\mathbf{SA}$, an algebra of situations $\mathcal{S}$ is therefore defined as follows:

Definition 4: $\mathcal{S} = \langle \mathbf{S}, \sqcup, \sqcap, -, \mathbf{1}, \mathbf{0} \rangle$ is a complete atomic Boolean algebra such that $\mathbf{SA} \subseteq \mathbf{S}$.

- $\mathbf{S}$, the universe of $\mathcal{S}$, is Wittgenstein's "logical space" (*logischer Raum*). The elements of $\mathbf{S}$ are called "situations" (*Sachlagen*) or "possible situations".
- "$\sqcup$" stands for "supremum" (least upper bound). "Totality" (*Gesamtheit*) is comprehensible if it is read as "supremum" up to TLP 3 and as "set" from then on.
- "$\sqcap$" stands for "infimum" (greatest lower bound), while "$-$" stands for "complement".
- $\mathbf{1}$ is the impossible situation, $\mathbf{0}$ the necessary situation. These are the two improper (*uneigentliche*) situations.
- A situation $\mathbf{s}$ "exists" (*besteht*) in a situation $\mathbf{s}'$ iff $\mathbf{s} \sqsubseteq \mathbf{s}'$. Synonyms: "$\mathbf{s}$ is the case (*ist der Fall*) at $\mathbf{s}'$", "$\mathbf{s}$ is contained (*ist enthalten*) in $\mathbf{s}'$".

---

[3]At this point, various "forms" may be introduced. In view of TLP 2.0141, the *Form des Gegenstandes* is: $FG(\mathbf{g}) = \{\langle i, \mathbf{g}_0 * \ldots * \mathbf{g}_i \rangle : n \geq i \text{ and } \mathbf{g} = \mathbf{g}_i\}$. A similar definition has been given by Mudersbach (1978). By Definition 3, clause (a), $FG(\mathbf{g}) \neq \emptyset$ (for any $\mathbf{g} \in \mathbf{G}$). This is Mudersbach's Axiom 5. The *Form des Sachverhaltes* is: $FS(\mathbf{g}_0 * \ldots * \mathbf{g}_n) = \{\langle i, FG(\mathbf{g}_i) \rangle : i \leq n\}$. Cf. the definition of the "structure" of $\mathbf{s}$ by Czermak (1979). The point which is made in the text is that $FG(\mathbf{g})$ and $FS(\mathbf{s})$ may be *proper* subsets of $\mathscr{P}(N \times \mathbf{G^*})$ and $\mathscr{P}(N \times \mathscr{P}(N \times \mathbf{G^*}))$, respectively (where "$\mathscr{P}$" denotes the power-set), but that it cannot be determined in advance *which* subsets they are. Forms are not *a priori*.

[4]Any Boolean algebra may be regarded as a mereology (theory of parts and wholes). A mereological view of the *Tractatus* has also been argued for by Simons (1986).

- A dual atom of $\mathbf{S}$ is called a "possible world" (*mögliche Welt*, NB 19.9.16; cf. TLP 2.022) or "world" (*Welt*, TLP *passim*) for short. The set of dual atoms will be denoted by "$\mathbf{W}$".[5]
- Some (arbitrary) element $\mathbf{w}_0 \in \mathbf{W}$ is "the" world, the "actual world" (*die wirkliche Welt*, TLP 2.022), "our world" (*unsere Welt*, TLP 6.1233), "the world in which we live" (*die Welt worin wir leben*, NB p. 127).[6]
- A situation "exists" or "is the case" *simpliciter* iff it exists (is the case) in $\mathbf{w}_0$.
- A "fact" (*Tatsache*) is a situation which is the case.[7]

Using the above informal paraphrases of our technical terms, some features of Tractarian situation ontology can already be given a precise interpretation. For example, it is an elementary thesis of Boolean algebra that $\mathbf{w}_0 = \bigsqcup\{\mathbf{s} \in \mathbf{S} : \mathbf{s} \sqsubseteq \mathbf{w}_0\}$; translation in informal terms yields "Die Welt ist die Gesamtheit der Tatsachen", i.e., TLP 1.1. More or less the same is stated in TLP 1 and 1.11–1.2, (which shows that the beginning of the *Tractatus* is rather repetitive, as Menger (1980) already pointed out in his plea for a formal analysis of the work).

1.3. **Complete Sets of States of Affairs.** A *complete* set of situations is a set which contains, for every situation, either this situation or its complement, but not both, and no other elements. We require that $\mathcal{S}$ satisfies the following additional conditions:

Condition 1: For every complete set of states of affairs $\mathbf{K}$: $\bigsqcup \mathbf{K} \neq \mathbf{1}$.

Condition 2: If $\mathbf{w}$, $\mathbf{w}' \in \mathbf{W}$ and $\mathbf{w} \neq \mathbf{w}'$, then there is at least one $\mathbf{s} \in \mathbf{SA}$ such that $\mathbf{s} \sqsubseteq \mathbf{w}$ and $\mathbf{s} \not\sqsubseteq \mathbf{w}'$.

Condition 1 states that all states of affairs are *independent* in the sense of Boolean algebra. This seems to be a good explication of the Tractarian thesis that the states of affairs are independent (*unabhängig*, TLP 2.061): it enables us to prove such passages as TLP 2.062 and 4.27, which have baffled many commentators. Consider 2.062 as an example (we shall discuss 4.27 in a moment). Here it is said that the existence of a state of affairs cannot be inferred from the existence of another state of affairs. This is easily provable: assume one *could* do the latter, i.e., that $\mathbf{s} \sqsubseteq \mathbf{s}'$ for some $\mathbf{s}$, $\mathbf{s}' \in \mathbf{SA}$. Then $\bigsqcup\{-\mathbf{s}, \mathbf{s}'\} = \mathbf{1}$, which contradicts Condition 1; hence the assumption is false, Q.E.D. As we shall see, all Tractarian remarks on the independence of elementary sentences may also be proved using Condition 1.

At this point, Condition 2 cannot yet be very well justified. However, it yields one nearly Tractarian thesis: "Die Gesamtheit der bestehenden Sachverhalte *bestimmt* die Welt"; this is TLP 2.04 upon emendation along the lines indicated by Griffin[8]. Moreover, there is indirect reason to accept Condition 2: as we shall see in Section 3.5, it has the consequence that each possible world is completely describable by elementary sentences, which is certainly a prominent thesis of the *Tractatus*.

By Boolean algebra, the conjunction of Conditions 1 and 2 is equivalent to:

---

[5] It is no anachronism to discuss the *Tractatus* in terms of modern "possible worlds": the latter notion was introduced in twentieth-century philosophy by Carnap (1956, p. 91), who took it in his turn directly from the *Tractatus*.

[6] The relation between the terms "world" and "reality" (*die Wirklichkeit*) is hard to understand. Czermak (1979) suggests interpreting "reality" as the partition $\langle\{\mathbf{s} \in \mathbf{SA} : \mathbf{s} \sqsubseteq \mathbf{w}_0\}, \{\mathbf{s} \in \mathbf{SA} : \mathbf{s} \not\sqsubseteq \mathbf{w}_0\}\rangle$ (cf. TLP 2.06, 4.0621); but "reality" may sometimes mean "logical space" or "the world" as well.

[7] However, "fact" is sometimes used in the sense of "concatenation" (see note 11); in this case the term may refer to a non-existent situation. In "Komplex und Tatsache" (ca. 1931; repr. in Wittgenstein (1964)) the requirement that facts exist is explicitly dropped.

[8] Griffin (1965, ch. 5). Emendation of TLP 2.04 is needed anyway, because it cannot be brought into line with TLP 2.06 and 2.063 otherwise. The emended version may be compared with TLP 2.05 and 4.26–4.28.

Condition 3: For every complete set of states of affairs $\mathbf{K}$: $\bigsqcup \mathbf{K} \in \mathbf{W}$.[9]

It follows from Condition 3 that $\mathbf{SA}$ is a set of generators of $\mathcal{S}$. Therefore, $\mathcal{S}$ may briefly be characterized as a *complete atomic Boolean algebra independently generated by* $\mathbf{SA}$. It follows that $Card(\mathbf{W}) = 2^{Card(\mathbf{SA})} = 2^{\aleph_0}$, while $Card(\mathbf{S}) = 2^{Card(\mathbf{W})}$, as has also been concluded by Suszko (1968, p. 21). The *Tractatus* explicitly mentions the finite analogue of the first property of $\mathcal{S}$: n states of affairs generate 2^n worlds (TLP 4.27). It does not mention the second property.

1.4. **Summary.** Recapitulating this section, we say that a *Tractarian ontological system* is a quadruple $\Sigma = \langle \mathbf{G}, \mathbf{SA}, \mathcal{S}, \mathbf{w}_0 \rangle$ satisfying Definitions 1–4 and Condition 3.

2. Syntax of Sentences, Thoughts and Pictures

2.1. **Syntax of Sentences.** The building-blocks of sentences (*Sätze*) are names (*Namen*: TLP 3.202, 3.26, 4.0311, 4.22, 5.55). The category of names is a very general one. For example, no explicit distinction is made between names of individuals (particulars) and predicates: predicates are simply names too, viz., names of properties and relations. As NB 31.5.15 says: "[Two] names are necessary for an assertion that *this* thing possesses *that* property". Therefore we shall not explicitly distinguish between, say, "proper" names (designating individuals) and predicates, but treat them on the same footing (cf. Section 1.1). As there is a bijection from $\boldsymbol{N}$ to some $\mathbf{G}$ (Definition 15b), the cardinality of $\boldsymbol{N}$ satisfies the same restrictions as that of $\mathbf{G}$. Thus (cf. Definition 1):

Definition 5: $\boldsymbol{N}$ is a set such that $1 \leq Card(\boldsymbol{N}) \leq \aleph_0$.

Just as states of affairs are concatenations of objects, so elementary sentences (*Elementarsätze*) are concatenations of names (TLP 4.22; cf. TLP 3.14, 3.21, 4.221). Designating the set of elementary sentences by "$\boldsymbol{EL}$" (for "elementary language"), we accordingly have (cf. Definitions 2, 3):

Definition 6: . $\boldsymbol{N*}$ is the smallest set such that:
(a) if $a, a' \in \boldsymbol{N}$ then $a * a' \in \boldsymbol{N*}$;
(b) if $a \in \boldsymbol{N}$ and $p \in \boldsymbol{N*}$, then $a * p \in \boldsymbol{N*}$.

Definition 7: $\boldsymbol{EL} \subseteq \boldsymbol{N*}$ is a set such that:
(a) For each $a \in \boldsymbol{N}$ there is a $p = a_0 * \ldots * a_n \in \boldsymbol{EL}$ such that $a = a_i$ for some i, $0 \leq i \leq n$, and
(b) $Card(\boldsymbol{EL}) = \aleph_0$.

The above definitions do not uniquely specify one set of elementary sentences; rather, they specify the broad conditions any such set must comply with. Because the structure of language reflects the structure of reality, syntax cannot be fully specified *a priori*. As TLP 5.55 says: "Since [...] we are unable to give the number of names with different meanings, we are also unable to give the composition of elementary sentences".

Suppose, for example, that there are both individuals (particulars) and properties. Then we may distinguish between "proper" names (names of objects of the former kind) and predicates (names of objects of the latter kind); but in this case presumably not all concatenations of names will be well-formed elementary sentences, for not each concatenation will correspond to a similarly structured state of affairs. For example, if F is a predicate and a a proper name ($F, a \in \boldsymbol{N}$), then $F * a$ may well be an elementary sentence, but $a * F$ will presumably be as ill-formed as

[9]Proof: by Condition 2 it cannot be the case that there are two different $\mathbf{w}, \mathbf{w}' \in \mathbf{W}$ such that $\bigsqcup \mathbf{K} \sqsubseteq \mathbf{w}$ and $\bigsqcup \mathbf{K} \sqsubseteq \mathbf{w}'$ (where $\mathbf{K}$ is a complete set of states of affairs). Therefore $\bigsqcup \mathbf{K} \in \mathbf{W}$ or $\bigsqcup \mathbf{K} = \mathbf{1}$; hence $\bigsqcup \mathbf{K} \in \mathbf{W}$ by Condition 1. That Condition 3 implies Conditions 1 and 2 is obvious.

the situation $\mathbf{g} * \mathbf{f}$ from Section 1.1. (This is not to say that predicates cannot be predicated in turn: we may have $G*F$ ($G, F \in \boldsymbol{N}$), where G is a second-order predicate.) Thus, the syntax of elementary sentences parallels the structure of states of affairs. Syntactical form mirrors ontological form.[10] As the latter is not *a priori* determinable, the former is not either.

According to the *Tractatus*, all sentences are built from the elementary sentences by means of the operation of joint negation N (TLP 5.5–5.51, 5.52, 6.001). Much fuss has been made over this operator, especially in connection with the Tractarian account of quantification; a good discussion is Soames (1983). Our solution will be simpler than Soames's in that we shall allow arbitrary *countable* sets of sentences as arguments for joint negation. Unfortunately, this does not agree well with TLP 5.32, where it is asserted that "All truth-functions are results of the successive application to elementary sentences of a finite number of truth-operations" (where "truth-operation" means "connective": see our discussion of TLP 5.54 in Section 4.1 below). However, our solution not only makes it possible to define quantification in terms of joint negation, it also enables us to express the independence of the elementary sentences and the principle of truth-functionality within our language (see Sections 3.4–3.6). So let us ignore TLP 5.32 and run the risk of making the *Tractatus* more interesting than it actually is!

In order to formulate the just-mentioned principles, we additionally need one other operator which is not *expressis verbis* to be found in the *Tractatus*: the unary modal connective $\Box$ (for "it is necessary that").

Thus, given some set $\boldsymbol{EL}$, the language $\boldsymbol{L}$ is defined as follows:

Definition 8: . $\boldsymbol{L}$ is the smallest set such that:
- (a) $\boldsymbol{EL} \subseteq \boldsymbol{L}$,
- (b) if $P \subseteq \boldsymbol{L}$, then $NP \in \boldsymbol{L}$, provided $1 \leq Card(P) \leq \aleph_0$,
- (c) if $p \in \boldsymbol{L}$, then $\Box p \in \boldsymbol{L}$, and
- (d) If $p \in \boldsymbol{L}$, then $(x)px \in \boldsymbol{L}$, where px is like p except that at least one occurrence of some name occurring in p has been replaced by x.

Negation, countable conjunction, the connective $\Diamond$ (for "it is possible that") and the universal quantifier (x) may be defined as follows (therefore the clause (d) was in fact superfluous):

- $\neg p = N\{p\}$.
- $\bigwedge P = N\{\neg p : p \in P\}$.
- $\Diamond p = \neg\Box\neg p$.
- $(x)px = \bigwedge\{p[a/x] : a \in \boldsymbol{N}\}$, where $p[a/x]$ is like px except that all free occurrences of x in px have been replaced by a.

Notice that variables may range over objects, properties, second-order properties, etc.; we cannot settle *a priori* what they range over since language depends on ontology. Therefore, we do not know *a priori* of what order Tractarian logic is; we only know that its order must be smaller than $Card(\boldsymbol{N})$. (It does not have to be 2, as Skyrms (1981) supposes.)

This completes the description of the construction of the ideal logical language $\boldsymbol{L}$ out of $\boldsymbol{EL}$. It will be seen that $\boldsymbol{L}$ does not contain an identity-sign, which is as it should be, for this is explicitly forbidden in TLP 5.53–5.5352 (see Section 3.1).

---

[10]Various "syntactical forms" may be defined in the same way as the ontological forms of note 3. The *Form des Bildelementes* FB of an element $e \in \boldsymbol{E}$ [$\boldsymbol{E} \supseteq \boldsymbol{N}$; see Section 2.3] is wholly analogous to FG, and the *Form der Abbildung* FA of an element $b \in \boldsymbol{EB}$ [$\boldsymbol{EB} \supseteq \boldsymbol{EL}$; see Section 2.3] is wholly analogous to FS. Similarly to $FG(\mathbf{g})$, $FB(e) \neq \emptyset$ by def. 7a; but for the rest, FB and FA are no more *a priori* than FG and FS are. The *Form der Darstellung* or *Form des Zusammenhangs* is the same as the *abbildende Beziehung*, which is defined in Section 3.2.

2.2. **Syntax of Thoughts.** Thoughts (*Gedanken*) are similar to sentences: "Thinking is a kind of language. [. . .] A thought is a kind of sentence" (NB 12.9.1916; cf. TLP 4). Analogously to sentences, thoughts are constructed from "psychical constituents that have the same sort of relation to reality as words" (NB p. 130). Denoting the set of thought-elements ("mental names") by $\boldsymbol{TE}$ and the set of elementary thoughts by $\boldsymbol{ET}$, we therefore stipulate that (cf. Definitions 5–7):

Definition 9: $\boldsymbol{TE}$ is a set such that $Card(\boldsymbol{TE}) \leq \aleph_0$.

Definition 10: $\boldsymbol{TE^*}$ is the smallest set such that:
(a) if $e, e' \in \boldsymbol{TE}$ then $e * e' \in \boldsymbol{TE^*}$;
(b) if $e \in \boldsymbol{TE}$ and $t \in \boldsymbol{TE^*}$, then $e * t \in \boldsymbol{TE^*}$.

Definition 11: $\boldsymbol{ET} \subseteq \boldsymbol{TE^*}$ is a set such that:
(a) For each $e \in \boldsymbol{TE}$ there is a $t = e_0 * \ldots * e_n \in \boldsymbol{ET}$ such that $e = e_i$ for some i, $0 \leq i \leq n$, and
(b) $Card(\boldsymbol{ET}) = \aleph_0$.

Given $\boldsymbol{ET}$, the set of thoughts ("language of thought") $\boldsymbol{T}$ is constructed as follows (cf. Definition 8):

Definition 12: $\boldsymbol{T}$ is the smallest set such that:
(a) $\boldsymbol{ET} \subseteq \boldsymbol{T}$,
(b) if $T \subseteq \boldsymbol{T}$, then $NT \in \boldsymbol{T}$, provided $1 \leq Card(\boldsymbol{T}) \leq \aleph_0$,
(c) if $t \in \boldsymbol{T}$, then $\Box t \in \boldsymbol{T}$, and
(d) if $t \in \boldsymbol{T}$, then $(y)ty \in \boldsymbol{T}$, where ty is like t except that at least one occurrence of some thought-element occurring in t has been replaced by y.

We shall return to thoughts in Sections 4.2 ff.

2.3. **Syntax of Pictures.** As the *Tractatus* leaves us completely in the dark with regard to the structure of pictures in general, we shall consider no other pictures than sentences and thoughts. The sets $\boldsymbol{E}$, $\boldsymbol{EB}$ and $\boldsymbol{B}$ of pictorial elements (*Bildelemente*, TLP 2.1514), elementary pictures (*Elementarbilder*, a term not to be found in the *Tractatus*), and pictures (*Bilder*), respectively, are therefore defined as follows:

Definition 13: $\boldsymbol{E}$, $\boldsymbol{EB}$ and $\boldsymbol{B}$ are sets such that $\boldsymbol{E} = \boldsymbol{N} \cup \boldsymbol{TE}$, $\boldsymbol{EB} = \boldsymbol{EL} \cup \boldsymbol{ET}$, and $\boldsymbol{B} = \boldsymbol{L} \cup \boldsymbol{T}$.

2.4. **Summary.** Recapitulating this section, we say that a *Tractarian pictorial system* is a 9-tuple $\Pi = \langle \boldsymbol{E}, \boldsymbol{TE}, \boldsymbol{N}, \boldsymbol{EB}, \boldsymbol{ET}, \boldsymbol{EL}, \boldsymbol{B}, \boldsymbol{T}, \boldsymbol{L} \rangle$ satisfying Definitions 5–13 above.

3. Pictorial and Linguistic Representation

3.1. **Basic Picture Theory.**

Definition 14: A Tractarian interpretation for a pictorial system

$$\Pi = \langle \boldsymbol{E}, \boldsymbol{TE}, \boldsymbol{N}, \boldsymbol{EB}, \boldsymbol{ET}, \boldsymbol{EL}, \boldsymbol{B}, \boldsymbol{T}, \boldsymbol{L} \rangle$$

as described in Section 2.4 is a pair $I = \langle \Sigma, \delta \rangle$ such that:
(a) $\Sigma = \langle \mathbf{G}, \mathbf{SA}, \mathcal{S}, \mathbf{w}_0 \rangle$ is a Tractarian ontological system as described in Section 1.4;
(b) $\delta : \boldsymbol{E} \mapsto \mathbf{G}$ is a function such that $\delta \restriction \boldsymbol{N}$ is a bijection; and
(c) $\boldsymbol{B} \subseteq \mathbf{S}$.

The above provides the basis of the picture theory. $\delta(e)$ is the denotation (*Bedeutung*, "meaning" in NB) of e. When $\mathbf{g} = \delta(e)$, we say that e denotes or stands for (*steht für*, *bedeutet*, *vertritt*) $\mathbf{g}$ (TLP 3.203–3.221, 3.323, 4.0311, 4.0312). In this case, $\mathbf{g}$ is the object corresponding to or correlated with the pictorial element e

(*der dem Bildelement entsprechende, zugeordnete Gegenstand*: cf. TLP 2.13, 2.1514, 5.526). δ is *not* a function of the *Sachlage* $\mathbf{s}$ under consideration: Tractarian names are rigid designators (cf. Cocchiarella (1984), Soames (1983)).

In clause (b), "$\delta \restriction \boldsymbol{N}$ is a bijection" means that every object $\mathbf{g} \in \mathbf{G}$ has precisely one name. This follows from:

(i) $\delta \restriction \boldsymbol{N}$ is a surjection, i.e., for every $\mathbf{g} \in \mathbf{G}$ there is at least one $a \in \boldsymbol{N}$ such that $\mathbf{g} = \delta(a)$; otherwise there would be unnamed objects and hence indescribable situations, which contradicts TLP 4.26 (see Section 3.5).

(ii) $\delta \restriction \boldsymbol{N}$ is an injection, i.e., for every $\mathbf{g} \in \mathbf{G}$ there is at most one $a \in \boldsymbol{N}$ such that $\mathbf{g} = \delta(a)$. This is Wittgenstein's famous identity-theory, clearly expressed in TLP 5.53: "Identity of object I express by identity of sign, and not by using a sign for identity. Difference of objects I express by difference of signs." (See TLP 5.53–5.5352.)

Clause (c) expresses the Tractarian thesis that every picture is a situation.[11] Some situations may be regarded from two different points of view: they may be regarded as situations in their own right (in which case they will appear as, e.g., concatenations of objects), or they may be regarded as pictures (in which case they will appear as, e.g., concatenations of pictorial elements). There is no conflict between these two perspectives; the identification of pictures with situations is inconsequential from a semantical point of view.[12]

3.2. **Senses.** On the basis of δ a function $\sigma : \boldsymbol{B} \mapsto \mathbf{S}$ is defined as follows:

Definition 15: $\sigma : \boldsymbol{B} \mapsto \mathbf{S}$ is a function such that:

(a) If $e = e_0 * \ldots * e_n \in \boldsymbol{EB}$, then $\sigma(e) = \delta(e_0) * \ldots * \delta(e_n)$,

(b) $\sigma(NP) = \bigsqcup\{-\sigma(b) : b \in P\}$ (where $P \subseteq \boldsymbol{B}$), and

(c) $\sigma(\Box b) = \mathbf{0}$ if $\sigma(b) = \mathbf{0}$, $\sigma(\Box b) = \mathbf{1}$ otherwise (where $b \in \boldsymbol{B}$).

σ(b) is the sense (*Sinn*) of b ("Das Bild stellt eine mögliche Sachlage im logischen Raume [...] dar", TLP 2.202; cf. 2.11, 2.221). When $\mathbf{s} = \sigma(b)$, we say that $\mathbf{s}$ is "represented" (*dargestellt*, *abgebildet*) by b, that b "shows" (*zeigt*) $\mathbf{s}$, and that "b says (that) $\mathbf{s}$ is the case" (*b sagt, daß $\boldsymbol{s}$ der Fall ist*). As TLP 4.022 says: "The sentence *shows* how things stand *if* it is true. And it *says that* they do so stand". σ itself may be called the "pictorial relationship" (*abbildende, darstellende Beziehung*, TLP 2.1513–4).

Clause (a) is a succinct formulation of the picture theory for elementary pictures (cf. TLP 2.15, 3.1432, 4.0311, 4.21). As TLP 2.1514 says, in the case of elementary pictures, "the *abbildende Beziehung* [σ] consists of the correlations [*Zuordnungen*] of the picture's elements with objects". Supposing that $e_0 = R$ is an n-ary predicate and $\mathbf{g}_0 = \mathbf{R}$ an n-ary relation, we see that if $\mathbf{g}_i = \delta(e_i)$ for i, $0 \leq i \leq n$, then the R-configuration of $e_1, \ldots, e_n$ represents the $\mathbf{R}$-configuration of $\mathbf{g}_1, \ldots, \mathbf{g}_n$. Or to put it differently, let $R' = \{\langle e_i, \ldots, e_{i+n}\rangle \in \boldsymbol{N}^n : R * e_i * \ldots * e_{i+n} \in \boldsymbol{EB}\}$ and $\mathbf{R}' = \{\langle \mathbf{g}_i, \ldots, \mathbf{g}_{i+1}\rangle \in \mathbf{G}^n : \mathbf{R} * \mathbf{g}_i * \ldots * \mathbf{g}_{i+1} \in \mathbf{SA}\}$: then the "fact" that $e_i, \ldots, e_{i+n}$ stand in the relation R' says that $\mathbf{g}_i, \ldots, \mathbf{g}_{i+1}$ stand in the relation $\mathbf{R}'$—which is exactly what the notorious TLP 3.1432 affirms. In conjunction with

[11] Actually, TLP 2.141 says that every picture is a *fact* (this is repeated in 2.14 for sentences). However, requiring every picture to be an existing situation would seem unduly stringent. In our opinion, 2.14 and 2.141 primarily draw attention to the similarity of structure between states of affairs and elementary pictures: both are concatenations (of objects and pictorial elements, respectively). These passages do not expressly mean to say that every picture exists as part of $\mathbf{w}_0$. We could require the latter, but as nothing seems to be gained by this we take *Tatsache* here as *mögliche Tatsache* (cf. note 7).

[12] In order to clarify matters further we might introduce a function π mapping situations-as-situations onto situations-as-pictures, and a function π' mapping situations-as-pictures into situations-as-situations (cf. Favrholdt (1964)). π is not an injection, for different situations may be the same from the pictorial point of view: "'A' is the same sign as 'A'" (TLP 3.203).

Definition 14b, Definition 15a implies that there is a 1-1 correspondence between elementary sentences and states of affairs (i.e., $\sigma \restriction \boldsymbol{EL} \mapsto \mathbf{SA}$ is a bijection).

Clause (b) is equivalent to the following two claims taken together:

(i) $\sigma(\neg b) = -\sigma(b)$, and
(ii) $\sigma(\bigwedge P) = \bigsqcup\{\sigma(b) : b \in P\}$.

Both formulae are in a general way justified by TLP 5.2341: "The sense of a truth-function of p is a function of the sense of p". More specifically, (i) is justified by TLP 5.2341 ("Negation reverses the sense of the sentence") and by TLP 4.0621 ("The sentences p and $\neg p$ have opposite sense"); it also explains why p and $\neg\neg p$ have the same sense (TLP 4.0621). Justification for (ii) is harder to find; the *Tractatus* is silent on the semantics of conjunctions. However, NB 5.6.15 asserts that "$p \wedge \neg p$ is that thing [...] that p and $\neg p$ have in common". According to our formalization, the sense of $\bigwedge P$ is, indeed, the greatest common part (infimum) of the senses of the elements of P, which seems an acceptable way to give this assertion its due. It will be noted that if $p \notin \boldsymbol{EL}$, then $\sigma(p)$ is not a *new* element of $\mathbf{S}$ (if $\sigma(p) = \mathbf{s}$, $\mathbf{s}$ already belonged to $\mathbf{S}$), which is precisely what is asserted in TLP 3.42.

Because the senses of contradictions and tautologies are improper situations (for $\sigma(p \wedge \neg p) = \sigma(p) \sqcup -\sigma(p) = \mathbf{1}$, while $\sigma(p \vee \neg p) = \sigma(p) \sqcap -\sigma(p) = \mathbf{0}$), they are improper pictures themselves (cf. TLP 4.462) and they may even be called "senseless" (*sinnlos*, TLP 4.461). But as TLP 4.4611 emphasizes, contradictions and tautologies are not "nonsensical" (*unsinnig*): this is a term reserved for pseudosentences, i.e., for sentence-like entities which do not belong to $\boldsymbol{L}$ at all. For example, the metalinguistic assertions of "the ladder-language" in which the *Tractatus* discusses object language are *unsinnig*—which implies that the whole *Tractatus* is *unsinnig* (a conclusion which is, indeed, drawn in TLP 6.54).

Clause (c) introduces an $S5$-like semantical analysis of modal sentences. We adopt the analysis by von Wright (1982 *a*), which singles out $S5$ as the correct formalization of the notion of modality in the *Tractatus*. Von Wright's analysis is not without its critics; for example, Perzanowski (1985) regards several other modal logics as more suitable for this role. However, a strong point in favour of $S5$ is that its semantics make clear (as the semantics of other modal logics do not) why modal sentences do not violate the principle that the world is completely describable by elementary sentences (Section 3.5). As a consequence these sentences do not violate the principle of truth-functionality either (Section 3.6).[13] In view of the important role of both principles in the *Tractatus*, $S5$ seems to be the modal logic which agrees best with the *Tractatus*.

3.3. **Truth.** A function $TV : \boldsymbol{B} \times \mathbf{S} \mapsto \{T, F\}$ assigning a truth-value T (true) or F (false) to b at $\mathbf{s}$ is defined as follows:

Definition 16: $TV(b, \mathbf{s}) = T$ if $\sigma(b) \sqsubseteq \mathbf{s}$; $TV(b, \mathbf{s}) = F$ otherwise.

A picture b is said to be true (*simpliciter*) iff it is true at $\mathbf{w}_0$. Thus, a picture is true iff its sense exists, or in other words, it is true iff it says that $\mathbf{s}$ is the case and $\mathbf{s}$ is indeed the case (cf. the definitions of "exists", "is the case" and "says that" in Section 1.2 and Section 3.2). A picture is said to be valid (in the interpretation under consideration) iff $TV(b, \mathbf{s}) = T$ for all $\mathbf{s} \in \mathbf{S}$ (in this interpretation).

Definition 16 is an extension of TLP 4.25 to pictures and situations in general: "If the elementary sentence is true the [corresponding] state of affairs exists; if the elementary sentence is false the [corresponding] state of affairs does not exist" (cf. TLP 4.21). It is evident that there is a 1-1 correspondence between senses and

[13]For this reason $S5$ is sometimes, rather confusingly, called an "extensional" modal logic, e.g., by Perzanowski (1985). We shall see that truth-functionality and extensionality must be sharply distinguished (Section 3.6).

partitions $\langle\{\mathbf{s} \in \mathbf{S} : TV(b, \mathbf{s}) = T\}, \{\mathbf{s} \in \mathbf{S} : TV(b, \mathbf{s}) \neq T\}\rangle$ of logical space $\mathbf{S}$, which is in eminent agreement with TLP 2.11, 2.201, 4.1 and also explains TLP 4.024: "To understand a sentence means to know what is the case if it is true". (For knowing $\sigma(p)$ amounts to knowing the corresponding partition.) Specifying the (actual) truth-value of a sentence serves to narrow down the range (*Spielraum*) the "logical place" (*logischer Ort*) of $\mathbf{w}_0$ may occupy (TLP 4.463, 5.5262). This function is not fulfilled by tautologies and contradictions. As these determine the partitions $\langle\mathbf{S}, \emptyset\rangle$ and $\langle\emptyset, \mathbf{S}\rangle$, respectively, the former leave the whole subset $\mathbf{W}$ of logical space $\mathbf{S}$ open to the world, while the latter leave no point for it at all (TLP 4.46–4.4611).

Some observations (provable by elementary Boolean algebra):

(a) $TV(e_0 * \ldots * e_n, \mathbf{s}) = T$ iff $\mathbf{g}_0 * \ldots * \mathbf{g}_n \sqsubseteq \mathbf{s}$, where $\mathbf{g}_i = \delta(e_i)$ for all i, $0 \leq i \leq n$ (see TLP 4.21). It is evident that this truth-condition is hardly "related to the Tarski-type truth-definition for atomic sentences" (the latter claim has been made by Hintikka & Hintikka (1983, p. 158)).
(b) $TV(\neg b, \mathbf{w}) = T$ iff $TV(b, \mathbf{w}) = F$. This holds for worlds, but *not* for all situations: we have $TV(b, \mathbf{s}) = T$ for all b iff $\mathbf{s} = \mathbf{1}$, while we may have $TV(b, \mathbf{s}) = TV(\neg b, \mathbf{s}) = F$ if $\mathbf{s} \notin \mathbf{W}$.
(c) $TV(\bigwedge P, \mathbf{s}) = T$ iff $TV(b, \mathbf{s}) = T$ for all $b \in P$, and similarly for universally quantified sentences (or thoughts).
(d) b is valid iff $\sigma(b) = \mathbf{0}$.
(e) $\Box b$ is valid if b is valid; otherwise $\neg\Box b$ is valid.

3.4. The Independence of Elementary Sentences.

The independence of states of affairs is reflected in the independence of elementary sentences. Let us call a set of sentences "independent" (*unabhängig*) if the situations described by these sentences are independent, and let a "state-description" be a complete set of elementary sentences, i.e., a set which contains for every elementary sentence either this sentence or its negation, but not both, and no other elements (Carnap 1956, p. 9). By definition of σ the independence of $\boldsymbol{EL}$ may be given the following expression, which is immediately provable by Condition 1 on $\mathcal{S}$ (Section 1.3):

Theorem 1: For every state-description $\boldsymbol{SD}$, $\Diamond \bigwedge \boldsymbol{SD}$ is valid.

Thus, all members of any state-description are compossible, just as all members of any complete set of states of affairs are compossible.

With Theorem 1, all Tractarian assertions on the independence of elementary sentences may be proved. For example, TLP 4.211 says that elementary sentences do not contradict each other (cf. TLP 6.3751). Indeed, suppose that $\sigma(p) = -\sigma(q)$ for some $p, q \in \boldsymbol{EL}$. Then $\sigma(p) \sqcup \sigma(q) = \mathbf{1}$, which contradicts Theorem 1. Similarly, TLP 5.134 says that elementary sentences cannot be deduced from each other (cf. TLP 2.062, already discussed in Section 1.3). Indeed, suppose p follows from q ($p, q \in \boldsymbol{EL}$). Then $\sigma(p) \sqsubseteq \sigma(q)$, for as TLP 5.122 states, "If p follows from q, the sense of 'p' is contained in the sense of 'q'" (cf. the definition of "is contained in" in Section 1.2). It follows that $\sigma(p) \sqcup -\sigma(q) = \mathbf{1}$, which contradicts Theorem 1.

3.5. The Complete Describability of the World by Elementary Sentences.

TLP 4.26 asserts:

> If all true elementary sentences are given, the result is a complete description of the world. The world is completely described by giving all elementary sentences, and adding which of them are true and which false.

In our formal reconstruction, this follows immediately from Condition 2 on $\mathcal{S}$ (Section 1.3) and Definitions 15 and 16 of σ and TV:

Theorem 2: If $\mathbf{w} \neq \mathbf{w}'$ then there is at least one $p \in \boldsymbol{EL}$ such that $TV(p, \mathbf{w}) = T$ while $TV(p, \mathbf{w}') = F$.

Because two worlds cannot contain precisely the same states of affairs, they cannot agree on all elementary sentences (cf. the discussion of TLP 2.04 in Section 1.3); if two worlds differ, there is at least one elementary sentence describing the difference.

Theorems 1 and 2 are in conjunction equivalent to:

Theorem 3: $\sigma(\bigwedge \boldsymbol{SD}) \in \mathbf{W}$ for every state-description $\boldsymbol{SD}$,

which may also be directly derived from Condition 3 on $\mathcal{S}$ (Section 1.3). There is a 1-1 correspondence between worlds, the state-descriptions describing them, and the states of affairs existing in them: each state-description describes precisely one world, and each world is completely described by one state-description. (Therefore state-descriptions are in fact *world*-descriptions.) Non-elementary sentences such as quantified sentences are superfluous as far as the description of the world is concerned. This also applies to modal sentences: because their truth-value is the same in all possible worlds, they do not contribute to the description of any one world in particular.

Theorem 3 has an interesting consequence: for any interpretation and any $p \in \boldsymbol{EL}$, $\bigwedge \boldsymbol{SD} \supset p$ is valid iff $p \in \boldsymbol{SD}$.[14] Notice that this is a definition of validity for elementary sentences *which does not depend on the interpretation*. However, the set of all valid sentences $p \in \boldsymbol{L}$ is recursively definable in terms of the set of all valid elementary sentences, for we have:

Theorem 4: For any interpretation and any state-description $\boldsymbol{SD}$:

(a) For any $p \in \boldsymbol{EL}$: $\bigwedge \boldsymbol{SD} \supset p$ is valid iff $p \in \boldsymbol{SD}$.
(b) For any $p \in \boldsymbol{L}$: $\bigwedge \boldsymbol{SD} \supset \neg p$ is valid iff $\bigwedge \boldsymbol{SD} \supset$ p is not.
(c) For any $P \subseteq \boldsymbol{L}$: $\bigwedge \boldsymbol{SD} \supset \bigwedge P$ is valid iff $\bigwedge \boldsymbol{SD} \supset p$ is valid for all $p \in P$.
(d) For any $p \in \boldsymbol{L}$: p is valid iff $\bigwedge \boldsymbol{SD} \supset p$ is valid for all $\boldsymbol{SD}$.
(e) $\Box p$ is valid if p is valid; otherwise $\neg\Box p$ is valid.[15]

Therefore the above provides a recursive definition of validity for all sentences $p \in \boldsymbol{L}$ which is *independent from interpretations*. It follows that:

Theorem 5: Exactly the same sentences $p \in \boldsymbol{L}$ are valid in all interpretations.

3.6. **The Principle of Truth-functionality.** The Tractarian principle of truth-functionality is weaker than might be expected. According to present-day definitions, a sentence cannot be truth-functional (i.e., a sentence cannot contain only truth-functional connectives) unless its truth-value is some function of the truth-values of the *subsentences* it contains and of the way it is built up from these; the truth-values of other sentences do not count (see, e.g., Humberstone (1986)).[16]

The Tractarian formulation of truth-functionality seems more liberal: "The sentence is a truth-function of the elementary sentences. (The elementary sentence is a truth-function of itself.) The elementary sentences are the truth-arguments of sentences" (TLP 5-5.01). It does not seem to be required here that only the truth-values of the elementary *subsentences* of a sentence matter as to its truth-value; the

---

[14] Proof. "$\Rightarrow$": suppose $\sigma(\bigwedge \boldsymbol{SD} \supset p) = \mathbf{0}$. Then $\sigma(p) \sqsubseteq \sigma(\bigwedge \boldsymbol{SD})$. As $\sigma(\bigwedge \boldsymbol{SD}) \in \mathbf{W}$ by Theorem 3, therefore $-\sigma(p) \not\sqsubseteq \sigma(\bigwedge \boldsymbol{SD})$; accordingly $\neg p \notin \boldsymbol{SD}$, whence $p \in \boldsymbol{SD}$, Q.E.D. "$\Leftarrow$": if $p \in \boldsymbol{SD}$, then obviously $\sigma(p) \sqsubseteq \bigsqcup\{\sigma(p) : p \in \boldsymbol{SD}\}$, whence $\sigma(\bigwedge \boldsymbol{SD} \supset p) = \mathbf{0}$, Q.E.D.

[15] Proof of (a) is easy. (b)–(e) are most easily proved by realizing that $\bigwedge \boldsymbol{SD} \supset p$ is valid iff p is true at $\sigma(\bigwedge \boldsymbol{SD})$ (for p is true at $\sigma(\bigwedge \boldsymbol{SD})$ iff $\sigma(p) \sqsubseteq \sigma(\bigwedge \boldsymbol{SD})$ iff $\sigma(\bigwedge \boldsymbol{SD} \supset p) = \mathbf{0}$), and then using observations (b)–(e) on truth from Section 3.3.

[16] The set of subsentences of a sentence is defined as $Sub(p) = \{p\}$ if $p \in \boldsymbol{EL}$, $Sub(NP) = \{NP\} \cup \bigcup\{Sub(p) : p \in P\}$ and $Sub(\Box p) = \{\Box p\} \cup Sub(p)$.

latter truth-value may as well be a function of the truth-values of *all* elementary sentences.

We shall take the Tractarian principle of truth-functionality to mean the latter. Thus, this principle asserts that the truth-values of the elementary sentences jointly determine the truth-values of all sentences; given a state-description, any sentence may assume only *one* truth-value. Or to put it formally: the principle asserts that for all $p \in \boldsymbol{L}$, the relation $\{\langle \boldsymbol{SD}, TV(p, \mathbf{s})\rangle : \boldsymbol{SD}$ is a state-description such that $TV(\bigwedge \boldsymbol{SD}, \mathbf{s}) = T\}$ is a *function*; that is, there are no situations $\mathbf{s}$, $\mathbf{s}'$ verifying the same state descriptions while simultaneously $TV(p, \mathbf{s}) \neq TV(p, \mathbf{s}')$ for some $p \in \boldsymbol{L}$. Intuitively, the principle of truth-functionality is presupposed by the principle of the complete describability of the world by elementary sentences: if the former principle did not hold, two worlds could verify the same elementary sentences and yet differ as regards the truth-value of some other sentence—and they would hence not be *completely* described by elementary sentences. This intuition is borne out by our explication: Theorem 2 (the completely describability thesis) implies the principle of truth-functionality (Theorem 6a). Summarizing the above and adding some refinements, we define:

Definition 17: Let $\mathscr{R}(\boldsymbol{EL}, p) = \{\langle \boldsymbol{SD}, TV(p, \mathbf{s})\rangle : \boldsymbol{SD}$ is a state-description such that $TV(\bigwedge \boldsymbol{SD}, \mathbf{s}) = T\}$.

(a) p is a *truth-function* of $\boldsymbol{EL}$ (in an interpretation I) iff $\mathscr{R}(\boldsymbol{EL}, p)$ is a function (in I).

(b) The *principle of truth-functionality* holds for $\boldsymbol{L}$ iff, for all $p \in \boldsymbol{L}$ and all I, $\mathscr{R}(\boldsymbol{EL}, p)$ is a function.

(c) p is a *determinate* truth-function of $\boldsymbol{EL}$ iff $\mathscr{R}(\boldsymbol{EL}, p)$ is the *same* function in all I.

(d) p is an *indeterminate* truth-function of $\boldsymbol{EL}$ iff $\mathscr{R}(\boldsymbol{EL}, p)$ is a function which *varies* with I.

For the moment, we do not need (d); we shall encounter indeterminate truth-functions in Section 4.6.

Theorem 6: (a) The principle of truth-functionality holds for $\boldsymbol{L}$.

(b) All sentences $p \in \boldsymbol{L}$ are determinate truth-functions of $\boldsymbol{EL}$ (under Tractarian interpretations of $\boldsymbol{L}$).

(c) $\{\langle TV \restriction (\boldsymbol{EL} \times \{\mathbf{w}\}), TV \restriction (\boldsymbol{L} \times \{\mathbf{w}\})\rangle : \mathbf{w} \in \mathbf{W}\}$ is a function.

(d) $\Box(\bigwedge \boldsymbol{SD} \supset p) \vee \Box(\bigwedge \boldsymbol{SD} \supset \neg p)$ is valid for all $\boldsymbol{SD}$ and all $p \in \boldsymbol{L}$.

(e) For all $\boldsymbol{SD}$ and all $p \in \boldsymbol{L}$: either $\bigwedge \boldsymbol{SD}$ and p, or $\bigwedge \boldsymbol{SD}$ and $\neg p$, are not compossible.

(f) For any p, $\boldsymbol{EL} \cup \{p\}$ is not independent.[17]

Sentences beginning with a modal operator do not form an exception to the principle of truth-functionality: their truth-values are *constant* functions of the truth-values of the elementary sentences. That is to say, $\mathscr{R}(\boldsymbol{EL}, \Box p)$ is a *constant* function: for all $\mathbf{s} \in \mathbf{S}$, $TV(\Box p, \mathbf{s})$ is the same. This also holds for $\mathscr{R}(\boldsymbol{EL}, \Diamond p)$.

As all logicians know (but most commentators of the *Tractatus* do not), the principle of truth-functionality must be distinguished from the principle of *extensionality*, which says that $(p \equiv p') \supset (q \equiv q')$ is valid in all interpretations, where q' is like q except that some occurrence of subsentence p of q has been replaced by p'. Because of the presence of modal sentences our Tractarian language is, though

[17]Proof of (a): by Theorem 2, if $TV(\bigwedge \boldsymbol{SD}, \mathbf{s}) = TV(\bigwedge \boldsymbol{SD}, \mathbf{s}') = T$, then $\mathbf{s} = \mathbf{s}'$ (and $\mathbf{s}, \mathbf{s}' \in \mathbf{W}$, or $\mathbf{s} = \mathbf{s}' = \mathbf{1}$); hence $TV(p, \mathbf{s}) = TV(p, \mathbf{s}')$ for any p, Q.E.D. Proof of (b): it must be shown that $\mathscr{R}(\boldsymbol{EL}, p)$ is the same function in all interpretations I, that is, that $TV(\bigwedge \boldsymbol{SD}, \mathbf{s})$ determines $TV(p, \mathbf{s})$ regardless of I. Proof: by Theorem 4, $TV(p, \mathbf{s}) = T$ just in case $\bigwedge \boldsymbol{SD} \supset p$ is valid; as validity does not depend on I by Theorem 5, $TV(\bigwedge \boldsymbol{SD}, \mathbf{s})$ determines $TV(p, \mathbf{s})$ regardless of I, Q.E.D. (c)–(f) are obvious consequences of (a).

truth-functional, definitely *not* extensional (see Humberstone (1986) for extensionality without truth-functionality).

4. Propositional Attitude Ascriptions

4.1. **Syntax of Propositional Attitude Ascriptions.** Wittgenstein begins his discussion of the propositional attitude ascriptions (from now on "thought-ascriptions" for short) by stating that "In the general sentential form a sentence occurs within a sentence only as a basis of truth-operations" (TLP 5.54). This is generally seen as an affirmation of the thesis of extensionality (Section 3.6), for instance by Black (1964).

However, this interpretation seems dubious. In TLP 5.542 Wittgenstein goes on to declare that thought-ascriptions are (appearances notwithstanding: TLP 5.541) no exception to the principle of TLP 5.54, because "A thinks that φ" has the same form as "'φ' says φ".[18] However, not only is it clear that "'φ' says φ" is *not extensional at all* (so this analysis would be pointless if TLP 5.54 really expressed the principle of extensionality): what is more, any analysis according to which "φ" occurs extensionally in "A thinks that φ" would simply be ludicrous, for thought-ascriptions are plainly not extensional.

Must we then conclude that "A thinks that φ" does not contain an occurrence of a subordinate sentence "φ" at all (as has, for example, been done by Black (1964) and Fogelin (1976))? This would not sound very convincing either. Obviously "φ" occurs in *some* sense in the latter sentence (albeit not in an extensional one), just as it occurs in some sense in "'φ' says φ". Both statements would be impenetrable, structureless wholes otherwise, which they not only do not *seem* to be, but also conflicts with the Tractarian view that no sentence is an unstructured whole: "The sentence is articulated" (TLP 3.251). Therefore we propose an alternative, more lenient interpretation of TLP 5.54.

In our view TLP 5.54 says nothing more than that sentences always occur *as sentences* within other sentences, and never as anything else. So sentences never occur in sentences as *names* of sentences, as *names* of facts, as *collections* of sentential *constituents* (names), as *affixes* (TLP 5.02), as *facts*, or whatever: subsentences of sentences have the same status as sentences standing on their own. That is to say, semantically they are *descriptions of situations*, and syntactically they are possible *arguments of connectives* (bases of truth-operations); this is already clear in the case of, e.g., negations, but TLP 5.54 emphatically repeats this principle for all sentences.

Now what should have been demonstrated after TLP 5.541 is that thought-ascriptions are only an apparent violation of the principle just mentioned. However, such a demonstration can only be found in the *Notebooks*. There it is clearly stated that in "A thinks that φ", "φ" plays the same syntactical role as it does in "not φ" and "it is necessary that φ": in "A thinks that φ", "'φ' cannot be replaced

[18] We shall use the Greek character "φ" as an *abbreviation for a sentence of ordinary language* and rewrite all quotations from Wittgenstein accordingly. This is necessary in order to prevent confusion with the formal development, where we use "p" as the *name* of the sentence $p \in \boldsymbol{L}$, as is usual in formal logic. Thus, it is correct to say that the name of a sentence $p \in \boldsymbol{L}$ is "p", but it would be absurd to say that the name of φ is "φ". Instead, we must say that the name of "φ" is "'φ'". For example, it is absurd to say that "the sun is shining" ("φ") is the name of (the situation) the sun is shining (φ), but it is correct to say that "'the sun is shining'" ("'φ'") is the name of "the sun is shining". To put it crudely, "p" corresponds to "'φ'", p corresponds to "φ", and $\sigma(p)$ corresponds to φ. Confusion between the names of sentences, sentences, and the senses of sentences would be fatal in the contexts considered here.

The same distinctions apply to the ordinary-language name of the subject, "A", and the formal counterpart of this name, A (although confusion is less serious here). Therefore we use the Greek character "A" in the former case and the Roman character "A" in the latter case.

by a proper name" (NB p. 95; cf. NB p. 106), nor will it do "to mention only its [i.e., "φ" 's] constituents, or its constituents and form but not in the proper order" (NB p. 94). Instead, "the sentence itself must occur in the statement to the effect that it is thought" (*Ibid.*). The underlying reason is that in "A thinks that φ", "φ" plays the same *semantical* role as in "not φ" and "it is necessary that φ": it is a *description of a situation*, just as in these other cases. And a situation cannot be described by a name or a *Klasse von Namen* (TLP 3.144, 3.142).[19]

As we have said, such a demonstration of the compatibility of thought-ascriptions with TLP 5.54 is, however, not to be found in the *Tractatus*. There it is only remarked that "A thinks that φ" is comparable to "'φ' says φ" (TLP 5.542). This comparison seems most unfortunate: in the latter sentence "φ" *does* occur as a name ("'φ'" is a name of "φ"), so this sentence violates the principle of TLP 5.54 (as we have interpreted it) and can hardly serve to explain why "A thinks that φ" does *not* violate it. Moreover, "'φ' says φ" is a metalinguistic statement and as such *unsinnig* (Section 3.2): but it is utterly implausible to suppose that "A thinks that φ" is *unsinnig*. The latter locution seems to be a perfectly proper part of everyday language. But then what is the function of the comparison in TLP 5.542? In our view it only serves to clarify the semantical analysis of thought-ascriptions (as we shall see in the next section). Thus, TLP 5.542 ff. do not bear on TLP 5.54-5.541 at all, stylistic appearances to the contrary notwithstanding. The *Tractatus* contains a gap between 5.541 and 5.542 which must be filled up with remarks from the *Notebooks*.

Summing up our discussion thus far: "A thinks that φ" is a sentence in which another sentence, "φ", occurs as the "basis" (the argument) of a "truth-operator" (connective). The "truth-operator" in question is "A thinks that ... ", which is comparable to "not ... ", and "it is necessary that ... ". Parallelling this analysis on the syntactic side we introduce the unary connective D_A, which is syntactically analogous to $\neg$ and $\Box$, and which may be read as "A thinks that ... " (*A denkt, daß* ... ; however, any propositional attitude may be substituted here):

Definition 18: $\boldsymbol{L}_D$ is the smallest set such that $\boldsymbol{L} \subseteq \boldsymbol{L}_D$ and if $p \in \boldsymbol{L}_D$, then $D_A p \in \boldsymbol{L}_D$; $\boldsymbol{T}_D$ is the smallest set such that $\boldsymbol{T} \subseteq \boldsymbol{T}_D$ and if $p \in \boldsymbol{L}_D$, then $D_A p \in \boldsymbol{T}_D$; $\boldsymbol{B}_D = \boldsymbol{L}_D \cup \boldsymbol{T}_D$.[20]

4.2. **Informal Semantics of Propositional Attitude Ascriptions.** Something about the semantics of thought-ascriptions may already be gleaned from the above: in "A thinks that φ", "φ" is a *description of a situation*. Like any sentence, it is a description of the situation which is its *sense*; as the *Notes on Logic* state, "here a *sense*, not a meaning [*Bedeutung*] is concerned" (NB p. 106). This immediately explains why "φ" cannot be "a piece of nonsense" here (as any adequate theory of

[19] This is Wittgenstein's objection to Russell's "theory of judgment", in which the propositional attitude ascription "A judges that a and b are similar", is analysed as "$J\{A, a, b, \text{similarity}, xFy\}$", where "$xFy$" stands for the form "something and something have some relation" (Russell 1984, p. 117). Here we *do* have a "class of names" (with one illegitimate name at that: according to Wittgenstein only objects can be named, forms cannot; cf. NB p. 105). As a result, the contact with situations is lost. Any class containing the appropriate constituents may be taken as the argument of a judgment-ascription, without any regard for situations. For example, $\{\text{this table}, \text{the book}, \text{penholders}, xFy\}$ would qualify (at least according to Wittgenstein); hence "A judges that this table penholders the book" would be a well-formed judgment-ascription on Russell's account (NB p. 103). Wittgenstein regards this as absurd.

[20] The operator "D_A" stands for the "most general" propositional attitude. If we start with several propositional attitudes D_A^n, $n \leq \omega$, then $D_A p$ may be defined as $D_A p = \bigvee\{D_A^i p : 0 \leq i \leq n\}$ (cf. note 30).

thought-ascriptions must do: TLP 5.5422), for situations cannot be described by "pieces of nonsense".[21]

So the semantical role of the subordinate sentence of a thought-ascription is already clear. What about the subject of the ascription (referred to by "A") and his or her relation to the sense of "φ", which is "obviously not a relation in the ordinary sense" (NB p. 95)? It is here that TLP 5.542–5.5421 come in.

> It is clear [...] that "A believes that φ", "A thinks φ", "A says φ" are of the form "'φ' says φ": and this does not involve a correlation of a fact with an object, but rather the correlation of facts by means of the correlation of their objects. This shows too that there is no such thing as the soul—the subject, etc.,—as it is conceived in the superficial psychology of the present day. For a composite soul would no longer be a soul.

In order to understand this passage of "almost impenetrable obscurity" (Urmson 1956, p. 133), we first have to understand the statement "'φ' says φ". This is not too difficult. Wittgenstein generally uses the name of a sentence (i.e., the sentence within quotation-marks) to refer to the sentence (cf. TLP 5.12, 5.123, 5.1241, 5.1311, 5.152, 5.44, 5.512, 5.513), and the sentence itself to refer to the situation described by the sentence (e.g., in TLP 5.43). Therefore "'φ' says φ" is a specification of the situation which is described by "φ", that is, it is a specification of the sense of "φ". But this specification is rather uninformative, for in order to describe the sense of "φ", "'φ' says φ" uses "φ" itself. Therefore "'φ' says φ" means nothing more than that the sense of "φ" is the sense of "φ": the statement is a correct but not very informative specimen of sense-specification. (An example: let "φ" = "The sun is shining". Then we have: "'The sun is shining' says the sun is shining". Of course it does; but this does not tell us much about the situation the sun is shining.) The formal rendering of "'φ' says φ" will be clear, given our definition of "p says $\mathbf{s}$ is the case" as $\sigma(p) = \mathbf{s}$ in Section 3.2: this is simply $\sigma(p) = \sigma(p)$, which is, again, correct but rather uninformative.

Now according to TLP 5.542 "A thinks that φ" is analogous to "'φ' says φ". This might be taken to mean that the former sentence is an instance of sense-specification as well. However, this would make thought-ascriptions *unsinnig*, as sense-specifications are metalinguistic assertions not belonging to the language itself (this also holds for their formal counterparts: $\sigma(p) = \mathbf{s} \notin \boldsymbol{L}$). But as we have already said, it is implausible to suppose that thought-ascriptions are nonsensical.

The solution to this problem is to assume that TLP 5.542 gives a *semantical* analysis of thought-ascriptions. Not the thought-ascription itself but its *sense* or its *truth-condition* is in some way analogous to "'φ' says φ". That is to say, "A thinks that φ" describes a *situation* which is in some way similar to the latter sentence; it is true iff that situation is indeed the case. Formally: not $D_A p$, but $\sigma(D_A p)$ is analogous to $\sigma(p) = \sigma(p)$ ("p says that $\sigma(p)$ is the case"). $D_A p$ itself is analogous to $\Box(p \equiv p)$, which *says that* $\sigma(p) = \sigma(p)$, that is, which describes a situation involving sense-specification. In the same way as $\sigma(\Box(p \equiv p))$ does, the situation $\sigma(D_A p)$ must somehow involve sense-specification.

If this is correct, "A", in "A believes that φ", must refer to at least one picture representing a situation (otherwise there would be nothing to specify the sense of). This picture cannot be an object: objects are incapable of representing because

[21] Syntactically, Definition 18 already prevents "nonsensical" thought-ascriptions from being well-formed sentences. Since $\boldsymbol{L}_D$ is the *smallest* set satisfying Definitions 8 and 18, it may be proved that $D_A p \in \boldsymbol{L}_D$ implies that $p \in \boldsymbol{L}_D$. Thus, it is *unmöglich einen Unsinn zu urteilen* (TLP 5.5422), for if the subordinate sentence p of $D_A p$ were *unsinnig* while $D_A p$ itself were not, then we would have a nonsensical $p \in \boldsymbol{L}_D$, which is impossible (since the nonsensicality of p implies that $p \notin \boldsymbol{L}_D$: Section 3.2).

they are simple. Only situations have the "logical complexity" which is required to represent (complex) situations (cf. TLP 2.02, 2.021, 3.142, 3.144, 4.032–4.041). So "A" refers to at least one situation, viz. the representing picture; in TLP 5.542 this situation is called a "fact", to which another "fact", namely the fact (or situation) represented by the former fact, is correlated. This makes it clear that "A" cannot be a name, for names always refer to objects. "A" is a "pseudo-name".

This makes one wonder about the references of pseudo-names. What do they refer to, in order that they may refer to one or more pictures? The answer is provided by Russell's writings of the same period. According to Russell symbols like "A" ("incomplete symbols", as he calls them) do not refer to simple objects, but to certain "logical fictions", namely classes, or series, or series of classes (Russell 1918, p. 253). "Persons are fictions" as well (Russell 1922, p. xix); the person referred to by "A" is similarly a "series of events" or a "class of facts" (Russell 1927, p. 403, p. 405). "The names that we commonly use, like 'Socrates', are really abbreviations for descriptions, not only that, but what they describe are not particulars but complicated systems of classes or series" (Russell 1918, pp. 200–1). Note that incomplete symbols are *not* rigid designators (in contrast to genuine names): the classes of facts they refer to form time-dependent *series*. Therefore their references vary with time (or per situation).

Wittgenstein supplements Russell's view of persons, then, by stating that some of the situations ("events", "facts") constituting a person in a given situation may have a pictorial character. Ascribing the thought that φ to a person amounts to asserting that among the pictures in question there is at least one which represents or models the situation φ.

Summing up, the above leads to the following analysis of thought-ascriptions: "A thinks that φ" is true (in a situation) iff the class of facts referred to by "A" (in that situation) contains at least one picture that says that φ. The latter picture may be, of course, be called a "thought" in the sense of Section 2.2. Thus, "A thinks that φ" is true iff the subject referred to by "A" has a thought that says that φ. It is of course the latter part of the truth-condition (the "business part", as Anscombe (1959, p. 88) called it) that is similar to "'φ' says that φ"; this is where the sense-specification comes in. Or to put it differently, "A thinks that φ" says that A has some thought that says that φ: it is the *sense* of "A thinks that φ" that is similar to "'φ' says that φ". A thought-ascription partially describes a person by means of specifying the sense of one of his thoughts, where the latter is done by employing a subordinate sentence having the same sense as that thought. (Because the sense of the thought is specified by using *another* picture, the latter sense-specification is not as uninformative as that in "'φ' says φ", where the *same* sentence is used to indicate the sense.) "A thinks that φ" is not meta-linguistic itself; but our analysis clearly vindicates the assertion by Clark (1976, p. 81) that "In ascribing thoughts and perceptions we are, very nearly, saying meta-linguistic things".

In view of the above the Tractarian account of thought ascription rather surprisingly turns out to be practically literally identical to that of modern "language of thought" theorists.[22] This is no shortcoming of our analysis: it only says something about the influentiality of the *Tractatus*![23]

---

[22]See, e.g., Field (1978) and Harman (1973). A detailed comparison of Harman's and Wittgenstein's language of thought theories has been carried out by Berghel (1978).

[23]The influence of TLP 5.542 may even be traced back in Wittgenstein's own strikingly similar remarks about "expecting" in the *Philosophische Bemerkungen* (ca. 1929).

> Ist es nicht so, daß meine Theorie ganz darin ausgedrückt ist, daß der Sachverhalt, der die Erwartung von p befriedigt, durch den Satz p dargestellt wird? (Wittgenstein 1964, remark no. 25).

Before proceeding with the formal semantics of thought-ascriptions, it should be remarked that incomplete symbols should not really be admitted in a logically perfect language (this was repeatedly emphasized by Russell). We have made an exception to this by allowing "A" in D_A, but we shall see that allowing this operator does not increase the capacity of the language to describe the world anyway—which is all the more justification to exclude incomplete symbols from an ideal logical language!

4.3. **Formal Semantics of Propositional Attitude Ascriptions.** The above insights may directly be incorporated into our formal semantics.

Definition 19: A Tractarian interpretation for a doxastic[24] pictorial system $\Pi = \langle \boldsymbol{E}, \boldsymbol{TE}, \boldsymbol{N}, \boldsymbol{EB}, \boldsymbol{ET}, \boldsymbol{EL}, \boldsymbol{B}_D, \boldsymbol{T}_D, \boldsymbol{L}_D \rangle$ as described in Section 2.4 and Definition 18, is a triple $I = \langle \Sigma, \delta, \psi_A \rangle$ such that Σ and δ are as in Definition 14 and $\psi_A : \mathbf{S} \mapsto \mathscr{P}(\mathbf{S})$ is a function such that $\{\sigma(t) : t \in \boldsymbol{T} \cap \psi_A(\mathbf{s})$ and $\Delta(t) = n\} = \bigcap\{\{\sigma(t) : t \in \boldsymbol{T} \cap \psi_A(\mathbf{w})$ and $\Delta(t) = n\} : \mathbf{w} \sqsupseteq \mathbf{s}\}$.

Here $\mathscr{P}(\mathbf{S})$ is the power-set of $\mathbf{S}$. The set of situations $\psi_A(\mathbf{s})$ is the "pseudo-denotation" of the "pseudo-name" A at $\mathbf{s}$.[25] As remarked above, A does not refer rigidly: ψ_A is not a constant function. $\Delta(b)$ is the doxastic degree of b, a notion which is similar to the usual notion of modal degree: $\Delta(b) = 0$ if $b \in \boldsymbol{EB}$, $\Delta(NP) = \max\{\Delta(b) : b \in P\}$, $\Delta(\Box b) = \Delta(b)$, and $\Delta(D_A p) = \Delta(p) + 1$. Notice that $\Delta(b) = 0$ iff $b \in \boldsymbol{B}$. The conditions on Δ and ψ_A will be motivated in a moment.

Definition 15 of σ is extended as follows:

Definition 20: $\sigma : \boldsymbol{B}_D \mapsto \mathbf{S}$ is a function such that the conditions of Definition 15 hold and moreover $\sigma(D_A p) = \sqcap\{\mathbf{s} \in \mathbf{S} :$ there is a $t \in \boldsymbol{T} \cap \psi_A(\mathbf{s})$ such that $\sigma(t) = \sigma(p)$ and $\Delta(t) \leq \Delta(p)\}$.

As Definition 16 is kept unchanged, Definition 20 ensures that $TV(D_A p, \mathbf{s}) = T$ iff there is a $t \in \boldsymbol{T}$ such that $t \in \psi_A(\mathbf{s})$ and $\sigma(t) = \sigma(p)$ and $\Delta(t) \leq \Delta(p)$. This is precisely the truth-clause we arrived at in our informal analysis in Section 4.2, except that we have extended our informal account with the condition that $\Delta(t) \leq \Delta(p)$. We have done so because we want the definition of the sense of $D_A p$ to be an *explanatory analysis* of $D_A p$ at the same time. For this to be the case, the definition must be a reductive one, that is, one in which the sense of $D_A p$ does not ultimately rest on the senses of other thought-ascriptions. (Theorem 8 below shows that our definition is a reductive one.) Otherwise, it could, e.g., be the case that $TV(D_A p, \mathbf{s}) = T$ iff there is a $t \in \boldsymbol{T} \cap \psi_A(\mathbf{s})$ such that $\sigma(t) = \sigma(p)$ and $\Delta(t) > \Delta(p)$. In this case it could be possible that the sense of $D_A p$ rests on that of $D_A D_A p$, with the sense of the latter in turn resting on that of $D_A D_A D_A p$, etc.: this would be one of the most mystifying analyses of $D_A p$ ever put forward!

The special condition on ψ_A in Definition 19 only serves to bring the doxastic interpretations into line with the non-doxastic semantics of Section 3. Using this condition, we may generalize the remarks on the senses and truth-values of sentences $p \in \boldsymbol{L}$ in Section III (up to Theorem 4) to *all* sentences $p \in \boldsymbol{L}_D$ and prove such

Die Erwartung, der Gedanke, der Wunsch, etc., daß p eintreffen wird, nenne ich erst dann so, wenn diese Vorgänge die Multiplizität haben, die sich in p ausdrückt, erst dann also, wenn sie *artikuliert* sind. (*Ibid.*, remark no. 32).

[24]From Greek "*dokein*" ("to think"), "*doxastikos*" ("pertaining to mere opinion, as opposed to knowledge").

[25]We might, but shall not, impose the following additional conditions on ψ_A. First, $\bigsqcup \psi_A(\mathbf{s}) \sqsubseteq \mathbf{s}$. Secondly, for every $\mathbf{s}' \in \psi_A(\mathbf{s})$ there is a $p \in \boldsymbol{L}_D$ such that $\mathbf{s}' = \sigma(p)$. Thirdly, $\mathit{Card}(\psi_A(\mathbf{s})) \leq \aleph_0$. In this case, A would be locally definable: for each $\mathbf{s} \in \mathbf{S}$, there would always be a true conjunctive sentence $p_{\mathbf{s}}$ such that $\sigma(p_{\mathbf{s}}) = \psi_A(\mathbf{s})$. A similar procedure may be hinted at in TLP 3.24.

theorems as $\sigma(D_A p) = \sqcap\{\mathbf{w} \in \mathbf{W} : TV(D_A p, \mathbf{w}) = T\}$ and $TV(D_A p, \mathbf{s}) = T$ iff $TV(D_A p, \mathbf{w}) = T$ for all $\mathbf{w} \sqsupseteq \mathbf{s}$. This would be impossible otherwise.

In order to gain a clear insight in the logical properties of thought ascriptions, we shall give a discussion exactly parallelling Section 3.4–3.6 above.

4.4. **The Interdependence of Propositional Attitude Ascriptions.** On this subject we may be very brief: as, e.g., $D_A p \equiv D_A \neg\neg p$ and $D_A(p \wedge q) \equiv D_A(q \wedge p)$ are valid, thought-ascriptions are clearly interdependent (i.e., not independent). In this respect they are similar to modal sentences. Therefore, there are no doxastic elementary sentences (as is already clear from the fact that $D_A p$ is not a concatenation of names).

4.5. **The Supervenience of Propositional Attitude Ascriptions on Elementary Sentences.** Although there are no doxastic elementary sentences, this does not affect the capacity of elementary sentences to provide complete descriptions of all worlds. Condition 2 and Theorem 2 still hold; therefore thought-ascriptions are as redundant as modal sentences as far as the describability of worlds is concerned. They partially describe persons by means of specifying the senses of some of their thoughts, but persons are, just as the worlds they form part of, already completely described by elementary sentences.

In modern parlance, this is expressed by calling thought-descriptions *supervenient* on elementary sentences. As Haugeland (1982, p. 97) defines it:

> Two worlds in $\mathbf{W}$ are *discernible with* language $\boldsymbol{L}$ just in case there is a sentence of $\boldsymbol{L}$ which is true at one, and not at the other. [. . .] $\mathbf{K}$ *weakly supervenes on* $\boldsymbol{L}$ (relative to $\mathbf{W}$) just in case any two worlds in $\mathbf{W}$ discernible with $\mathbf{K}$ are discernible with $\boldsymbol{L}$.

Accordingly, $\boldsymbol{L}$, $\boldsymbol{L}_D$, $\boldsymbol{T}$, $\boldsymbol{T}_D$, $\boldsymbol{B}$ and $\boldsymbol{B}_D$ all weakly supervene on $\boldsymbol{EL}$.

However, not all of Section 3.5 applies to thought-ascriptions: Theorem 4 now no longer provides a definition of validity for *all* sentences. Sentences p such that $\Delta(p) \geq 1$ (i.e., sentences not belonging to $\boldsymbol{L}$) are not covered by it. Indeed, it is readily seen that the validity of thought-ascriptions cannot be recursively defined for all interpretations I because it may vary with I. For example, we may have $\psi_A(\mathbf{s}) = \emptyset$ (for all $\mathbf{s}$) in I and $\psi_A(\mathbf{s}) = \mathbf{S}$ (for all $\mathbf{s}$) in J, with the result that $\neg D_A p$ is valid in I but invalid in J and that $D_A p$ is valid in J but invalid in I (any p).

4.6. **The (Indeterminate) Truth-functionality of Propositional Attitude Ascriptions.** Because Theorem 2 still holds, clauses (a) and (c)–(f) of Theorem 6 hold under substitution of $\boldsymbol{L}_D$ for $\boldsymbol{L}$. Thus, the principle of truth-functionality holds: the truth-value of a thought-ascription is a function of the truth-values of the elementary sentences. However, clause (b) of Theorem 6 does not hold for $\boldsymbol{L}_D$: as validity may now vary from interpretation to interpretation, we have by Definition 17d:

Theorem 7: Thought-ascriptions are indeterminate truth-functions of $\boldsymbol{EL}$ (under Tractarian interpretations of $\boldsymbol{L}_D$).

Thus, $\boldsymbol{EL} \cup \{D_A p\}$ is not independent, although the specific form of the dependence varies from interpretation to interpretation; in any interpretation either $\neg\Diamond(\bigwedge \boldsymbol{SD} \wedge D_A p)$ or $\neg\Diamond(\bigwedge \boldsymbol{SD} \wedge \neg D_A p)$ is valid, although it depends on the interpretation which one of both is valid. Of course, any p such that $\Delta(p) \geq 1$ is an indeterminate truth-function of $\boldsymbol{EL}$ as well.

Because $D_A p$ is an indeterminate truth-function of $\boldsymbol{EL}$, it is not sufficient to know the truth-values of the elementary sentences in order to know whether $D_A p$ is true. The case here is similar to the case of the description of the world by elementary sentences. Each state-description uniquely describes one world. But in order to know exactly *which* world it describes, one has to know some semantical facts: in particular, one has to know $\delta \upharpoonright \boldsymbol{N}$—and this is all one has to know, for $\sigma(\bigwedge \boldsymbol{SD})$ is fully determined by $\delta \upharpoonright \boldsymbol{N}$. The same is true for $\mathscr{R}(\boldsymbol{EL}, D_A p)$, the function telling how the truth-value of $D_A p$ depends on the truth-values of the elementary sentences (Definition 17): $\mathscr{R}(\boldsymbol{EL}, D_A p)$ is *some* function, but in order to know *which* function it is, one has to know some semantical facts. In particular, one has to know $\delta \upharpoonright (\boldsymbol{N} \cup \boldsymbol{TE}_A)$ and ψ_A (knowing $\delta \upharpoonright \boldsymbol{N}$ does not suffice), where $\boldsymbol{TE}_A$ is the set of thought-elements occurring in the sentences of "A's language of thought" $\boldsymbol{T}_A = \boldsymbol{T} \cap \bigcup\{\psi_A(\mathbf{s}) : \mathbf{s} \in \mathbf{S}\}$. $\delta \upharpoonright (\boldsymbol{N} \cup \boldsymbol{TE}_A)$ and ψ_A are *all* one has to know in order to determine $\mathscr{R}(\boldsymbol{EL}, D_A p)$, for:

Theorem 8: $\sigma \upharpoonright (\boldsymbol{L}_D \cup \boldsymbol{T}_A)$, and hence $\mathscr{R}(\boldsymbol{EL}, p)$, are fully determined by $\delta \upharpoonright (\boldsymbol{N} \cup \boldsymbol{TE}_A)$ and ψ_A.[26]

Will it ever be possible to know the meanings of all elements of $\boldsymbol{N} \cup \boldsymbol{TE}_A$, and of "$A$"? Assuming that it is unlikely that we may ever know the meanings of more than a finite number of pictorial elements, this depends on the cardinality of $\boldsymbol{N} \cup \boldsymbol{TE}_A$; only the eventual finiteness of $\boldsymbol{N} \cup \boldsymbol{TE}_A$ (and hence of $\mathbf{G}$) would guarantee a humanly possible determinability of $\sigma(D_A p)$ on the basis of denotations.[27]

Because thought-ascriptions are *indeterminate* truth-functions of the elementary sentences, it may be objected that the truth-functional account we have offered is not really a very *illuminating* one. We do not deny this; but the extreme generality of the analysis may well be unavoidable. The assertion that the truth-values of thought-ascriptions cannot vary unless some other features (e.g., physical features ultimately describable by elementary sentences) of the world do seems hazardous enough as it is. When doing logic (philosophy), we can hardly venture beyond this; any other, more specific systematic relationships there may be between elementary sentences and thought-ascriptions may well be of an empirical (or at least partly empirical) nature and should therefore be settled by empirical science. As a consequence, the extreme generality of the truth-functional account (which parallels the extreme generality of modern psychophysical supervenience theories) is not a defect, but a point in favour of the Tractarian theory.

4.7. **Tractatus 5.542 Formalized.** In order to demonstrate the adequacy of our formalization, let us show in detail how it ties in with TLP 5.542 and related passages. First, let p be an elementary sentence, $p = a_0 * \ldots * a_n$, $n \in \mathbb{N}$, with $\sigma(p) = \mathbf{g}_0 * \ldots * \mathbf{g}_n$. Then we have $TV(D_A p, \mathbf{s}) = T$ iff there is a $t \in \boldsymbol{T} \cap \psi_A(\mathbf{s})$ such

[26]Proof: Let $\Delta(b) = n$ (where $b \in \boldsymbol{L}_D \cup \boldsymbol{T}_A$). By Definition 20 and the definition of $\Delta(b)$, $\sigma(b)$ only depends on ψ_A and $\sigma \upharpoonright \{b \in \boldsymbol{L}_D \cup \boldsymbol{T}_A : \Delta(b) = n-1\}$. Repeating this argument as many times as necessary shows that $\sigma(b)$ depends only on ψ_A and $\sigma \upharpoonright \{b \in \boldsymbol{L} \cup \boldsymbol{T}_A : \Delta(b) = 0\}$. Since the latter function only depends on $\delta \upharpoonright (\boldsymbol{N} \cup \boldsymbol{TE}_A)$, $\sigma(b)$ is determined by $\delta \upharpoonright (\boldsymbol{N} \cup \boldsymbol{TE}_A)$ and ψ_A. As $\mathscr{R}(\boldsymbol{EL}, p)$ is known as soon as $\sigma(\bigwedge \boldsymbol{SD})$ and $\sigma(p)$ are given for all $\boldsymbol{SD}$ and p, and $\sigma(\bigwedge \boldsymbol{SD})$ is determined by $\delta \upharpoonright \boldsymbol{N}$, $\mathscr{R}(\boldsymbol{EL}, p)$ is fully determined by $\delta \upharpoonright (\boldsymbol{N} \cup \boldsymbol{TE}_A)$ and ψ_A, Q.E.D.

[27]Even if $\boldsymbol{N} \cup \boldsymbol{TE}_A$ were finite, this would not guarantee the definability (reducibility) of thought-ascriptions in terms of elementary sentences. The latter would obtain in the two following two cases; however, both cases are ruled out by the *Tractatus*. First, if $\mathbf{SA}$ were finite, then $\mathcal{S}$ would be finitely generated by $\mathbf{SA}$. In this case there would be a $q \in \boldsymbol{L}$ such that $\Delta(q) = 0$ and $\sigma(q) = \sigma(D_A p)$, and $D_A p$ would accordingly be reducible to elementary sentences. However, the *Tractatus* assumes $\mathbf{SA}$ to be infinite (Section 1.1). Secondly, if we allowed uncountable disjunctions, with $\sigma(\bigvee P) = \bigsqcap\{\sigma(p) : p \in P\}$, then $\sigma(D_A p) = \bigsqcap\{\mathbf{w} \in \mathbf{W} : \mathbf{w} \sqsupseteq \sigma(D_A p)\} = \bigsqcap\{\sigma(\bigwedge \boldsymbol{SD}) : \sigma(\bigwedge \boldsymbol{SD}) \sqsupseteq \sigma(D_A p)\} = \sigma(\bigvee\{\bigwedge \boldsymbol{SD} : \bigwedge \boldsymbol{SD} \supset D_A p\})$, and we would, again, have explicit definability (of a totally uninformative sort). However, such disjunctions are even less Tractarian than countable conjunctions. (Notice that they would make definite descriptions definable:

$$q(\imath x)px = \bigvee\{q[a/(\imath x)px] \wedge p[a/x] \wedge \bigwedge\{\neg p[a'/x] : a' \in \boldsymbol{N} \setminus \{a\}\} : a \in \boldsymbol{N}\},$$

where $q(\imath x)px$ is introduced in the same way as $(x)px$ in clause (d) of Definition 8.)

that $\Delta(t) \leq \Delta(p)$ and $\sigma(t) = \mathbf{g}_0 * \ldots * \mathbf{g}_n$. One t that would qualify is an elementary thought $t = e_0 * \ldots * e_n$ such that $\delta(e_i) = \delta(a_i) = \mathbf{g}_i$ for all i, $0 \leq i \leq n$. It is clear that in this case we have a *darstellende Beziehung* between two situations, namely t and $\sigma(p)$, by means of *Zuordnungen* of their elements, for $\sigma(t) = \sigma(p)$ because $\delta(e_i) = \delta(a_i)$ for all i, $0 \leq i \leq n$ (cf. TLP 2.1514). Or to speak very crudely, we have here a correlation of two "facts" (in the sense of "concatenations of elements") by means of a correlation of their "objects" (in the sense of "elements": pictorial elements in the one case, objects *sensu stricto* in the other), which is precisely what TLP 5.542 says. A's thought that p is true if $\mathbf{g}_0 * \ldots * \mathbf{g}_n$ is a fact, and it is false otherwise.[28]

Now let us define A's "soul" (*Seele*) at $\mathbf{s}$ as $\boldsymbol{T}_A(s) = \boldsymbol{T} \cap \psi_A(\mathbf{s})$. Thus A's soul (at $\mathbf{s}$) consists of A's thoughts (at $\mathbf{s}$); it is the currently entertained subset ("theory") of A's language of thought $\boldsymbol{T}_A$. And let us define "logical multiplicity" (*logische Mannigfaltigkeit*, TLP 4.04–4.0412, 5.475) as follows: $\mathit{Mult}(e_0 * \ldots * e_n) = n + 1$, $\mathit{Mult}(NP) = \mathit{Mult}(P) = \max\{\mathit{Mult}(b) : b \in P\}$, $\mathit{Mult}(\Box b) = \mathit{Mult}(D_A b) = \mathit{Mult}(b)$. It will be clear that the logical multiplicity of any non-empty set of pictures is at least 2, even if there existed only one pictorial element (and object). Therefore as soon as A thinks anything at all at $\mathbf{s}$ (i.e., $D_A p$ is true at $\mathbf{s}$ for some p) $\mathit{Mult}(\boldsymbol{T}_A(\mathbf{s})) > 1$: "It is just as impossible that [the subject] should be a simple as that 'φ' should be" (NB p. 119; cf. TLP 5.5421). This makes it clear that Wittgenstein's contention that the soul is complex should definitely not be taken to mean that $\mathit{Card}(\boldsymbol{T}_A(\mathbf{s})) > 1$, as Hintikka (1958, p. 90) considered admissible. Nor should it be taken to mean that thoughts are not "combined" with each other in the soul, taking "combination" in the sense of "conjunction" (i.e., if $T \subseteq \boldsymbol{T}_A(\mathbf{s})$, then $\bigwedge T \in \boldsymbol{T}_A(\mathbf{s})$): it is possible that the soul is "unified" in the sense that it is closed under conjunction, and it is also possible that it is not. In the terminology of psychologists of the time, Wittgenstein asserts that the soul is not *einfach* (simple), while he does not commit himself on the question as to whether it is *einheitlich* (unified).[29]

Because D_A stands for all propositional attitudes, the remarks on perception in TLP 5.5423 can also be easily understood. If $\sigma(a_0 * \ldots * a_n) = \mathbf{g}_0 * \ldots * \mathbf{g}_n$, then to perceive that $a_0 * \ldots * a_n$ is not just to have some isolated psychical elements referring to $\mathbf{g}_0, \ldots, \mathbf{g}_n$ separately: instead, it is to have a thought saying that $\mathbf{g}_0, \ldots, \mathbf{g}_n$ "are related to one another in such and such a way" (TLP 5.5423). For example, it is to have a thought saying that $\mathbf{g}_0, \ldots, \mathbf{g}_n$ are concatenated, in this sequence, or (if $\mathbf{g}_0 = \mathbf{R}$) it is to have a thought representing the $\mathbf{R}$-configuration of $\mathbf{g}_1, \ldots, \mathbf{g}_n$ (to recall Suszko's terminology from Section 1.1). This explains why seeing that $a_0 * \ldots * a_n$ is different from, say, seeing that $a_n * \ldots * a_0$: "for we really see two different facts" in the two cases (TLP 5.5423).[30]

---

[28]Copi has reached essentially the same insights in his fine informal article on TLP 5.542 (Copi 1958).

[29]The soul we are talking about here is the "human soul, with which psychology deals" (TLP 5.641). This empirical soul must be carefully distinguished from the "metaphysical subject", which is *not* a part of the world (5.641), and *simple*, not complex (5.64).

[30]Two *prima facie* different semantical analyses may be given of the propositional attitudes $D^n_A p$, $n \leq \omega$, other than D_A (see note 20). First, we may distinguish between various subsets ("dialects") $\boldsymbol{T}^n_A$, $n \leq \omega$, of the language of thought $\boldsymbol{T}_A = \bigcup\{\boldsymbol{T}^i_A : 0 \leq i \leq n\}$: in this case, perceptions, judgments, memories, etc., are special kinds of thoughts. To perceive (judge, remember) that p is to have a perception (judgment, memory) that says that p, etc. Secondly, we may distinguish between various "compartments" or "faculties" $\boldsymbol{T}^n_A(\mathbf{s})$, $n \leq \omega$, of the soul $\boldsymbol{T}_A(\mathbf{s}) = \bigcup\{\boldsymbol{T}^i_A(\mathbf{s}) : 0 \leq i \leq n\}$, say the faculties of perception, judgment, memory, etc. In this case, to perceive that p is to have a thought saying that p in one's faculty of perception, etc. As mental faculties may be defined in terms of mental dialects and *vice versa* (for $\boldsymbol{T}^n_A(\mathbf{s}) =$

The above may suffice as a demonstration of the adequacy of our formalization. It will be seen that few if any mysteries remain. The only unsolved problem is a historical one: which psychologists did Wittgenstein accuse of superficiality in TLP 5.5421? A search of the literature reveals that all major psychologists of the period regarded the soul as *einheitlich* but definitely not *einfach*![31]

5. Tractarian Doxastic Modal Logic

Wittgenstein himself was hardly interested in axiomatization (TLP 5.132), so we shall not go too deeply into this subject either. However, axiomatizing gives us a clear picture of what the preceding results lead up to; therefore we here present the doxastic modal logic *DML* (for a given Tractarian language $\boldsymbol{L}_D$) corresponding to the Tractarian semantics (for $\boldsymbol{L}_D$) proposed above.

5.1. Axiomatization of *DML*.

Axiom 1: Every axiom of finitary propositional logic is an axiom.
Axiom 2: $\bigwedge P \supset p$, where $p \in P$.
Axiom 3: $\Box p \supset p$.
Axiom 4: $\Diamond \bigwedge \boldsymbol{SD}$, for every state-description $\boldsymbol{SD}$.
Axiom 5: $\Box(\bigwedge \boldsymbol{SD} \supset p) \vee \Box(\bigwedge \boldsymbol{SD} \supset \neg p)$, for every state-description $\boldsymbol{SD}$.
Axiom 6: $\Box(p \equiv q) \supset (D_A p \equiv D_A q)$.
Rule 1: If $\vdash p$ and $\vdash p \supset q$, then $\vdash q$.
Rule 2: If $\vdash p \supset q$, for all $q \in Q$, then $\vdash p \supset \bigwedge Q$.
Rule 3: If $\vdash p \supset q$, then $\vdash p \supset \Box q$, provided p is fully modalized (i.e., provided every elementary sentence in p occurs within the scope of a modal operator).[32]

Here $\vdash p$ means that p is derivable in *DML*, i.e., that there exists a countable sequence $p_0, \ldots, p_j, \ldots, p_k$ such that $p_k = p$ and for each $j \leq k$, p_j is either an axiom or is inferred from earlier formulas p_i, $i < j$, by a rule of inference.

Theorem 9: $\vdash p$ iff p is valid in all Tractarian interpretations of $\boldsymbol{L}_D$.[33]

$\boldsymbol{T}^n_A \cap \boldsymbol{T}_A(\mathbf{s})$ and $\boldsymbol{T}^n_A = \bigcup\{\boldsymbol{T}^n_A(\mathbf{s}) : \mathbf{s} \in \mathbf{S}\}$), both approaches are formally the same. Notice that $D_A p \equiv \bigvee\{D^i_A p : i \leq n\}$ is valid (cf. note 20).

[31] See, for example, the following quotations, from books which explain the concepts of *Einfachheit* and *Einheitlichkeit* at greater length than we have done:

> Unsere Untersuchung hat ergeben, daß, wo immer eine Seelentätigkeit besteht, eine gewisse Mannigfaltigkeit und Verwickelung vorhanden ist. Selbst in dem einfachsten Seelenzustande ist ein doppelter Gegenstand immanent gegenwärtig. [. . .] Aber der Mangel an Einfachheit war nicht ein Mangel an Einheit. (Brentano 1973, p. 221).
>
> *Einheit* [ist] der treffendere Ausdrück für die Natur der Seele [. . .] als *Einfachheit*. (Fechner 1860, p. 415).
>
> Woher schöpft man die Überzeugung, daß die Seele ein *einfaches* Wesen sei? [. . .] Wir [treffen] in dem Bewußtsein [. . .] eine Mannigfaltigkeit an, die auf eine Vielheit seiner Grundlage hinweist. [. . .] Nicht als einfaches Sein, sondern als geordnete Einheit vieler Elemente ist die Seele was Leibniz sie nannte: *ein Spiegel der Welt.* (Wundt 1874, pp. 862–3).

[32] See Hughes & Cresswell (1972, p. 127), where it is also shown that Axioms 1 and 3 and Rules 1 and 3 jointly constitute an axiomatization of $S5$.

[33] Proof. "⇒" (soundness): by calculation. "⇐" (completeness): suppose that not $\vdash p$. Then construct a "canonical model" as follows. $\mathbf{S}$ is the power-set of the set of maximally consistent sets of *DML*. $\sqcup$, $\sqcap$ and $-$ are set-theoretical intersection (*sic*), union and complementation, respectively; $\mathbf{1} = \emptyset$, $\mathbf{0} =$ the set of all maximally consistent sets. $\mathbf{O} = \boldsymbol{N}$, δ is identity, $\sigma(p) = \{P \subseteq \boldsymbol{L}_D : P$ is maximally consistent and $p \in P\}$, and $\psi_A(\mathbf{s}) = \{p : D_A p \in \bigcap \mathbf{s}\}$, where $\mathbf{s}$ is a set of maximally consistent sets. Pictures may be identified with sentences, and these may be identified with arbitrary elements of $\mathbf{S}$ in turn. The model defined in this way is a genuine Tractarian interpretation. Axioms 4 and 5 jointly guarantee that Condition 3 on $\mathcal{S}$ is satisfied; Axiom 6 guarantees that ψ_A and $\sigma(D_A p)$ satisfy Definitions 19 and 20. Since p may be shown to

Theorem 9 holds *regardless of the order of DML* (which contrasts with the general situation for higher-order logic). The reason is clear: Tractarian interpretations correspond to Henkin's *general* models of higher-order logic (which enable completeness proofs), rather than to the so-called "natural" models (which do not). (Cf. Skyrms (1981, pp. 203–5).)

Because of the presence of $\bigwedge$, *DML* is undecidable. Without $\bigwedge$, *DML* would conceivably be decidable. (Whether it would actually be decidable depends on $Card(\boldsymbol{N})$ and thus on $Card(\mathbf{G})$; cf. Soames (1983, p. 588).)

5.2. **Some Observations on *DML*.** *DML* is an extension of the familiar logical systems $S5$ and $L\omega_1 0$ (classical propositional logic with countable conjunctions). The distinctive non-doxastic axioms of *DML* are Axioms 4 and 5. Axiom 4 is the linguistic counterpart of the thesis of the independence of states of affairs (Section 3.4). This axiom implies that $\vdash \Diamond \bigwedge P$ for any finite $P \subseteq \boldsymbol{SD}$, which formula culminates Suszko's discussion of the independence of states of affairs (Suszko 1968, Axiom 8.16). Axiom 5 is the linguistic expression of the thesis of truth-functionality (Sections 3.6 and 4.6). Notice that the expressibility of these theses within $\boldsymbol{L}_D$ crucially depends on the presence of $\Box$ and $\bigwedge$.

There is one little problem involving Axioms 4 and 5: because of their presence one might hesitate to regard *DML* as a logic at all. According to some definitions, e.g., one given by Perzanowski (1985), a logic should be closed under substitution. But in Axioms 4 and 5 $\bigwedge \boldsymbol{SD}$ may *not* be replaced by any arbitrary sentence $q \in \boldsymbol{L}_D$. Indeed, closure under substitution would have rather unpleasant consequences here. In this case, Axiom 4 would imply $\Diamond(p \wedge \neg p)$ and as we have $\vdash \neg\Diamond(p \wedge \neg p)$ thus bring about the inconsistency of *DML*. On the other hand, Axiom 5 would in this case imply $\Box((p \vee \neg p) \supset q) \vee \Box((p \vee \neg p) \supset \neg q)$, whence $\Box q \vee \Box \neg q$, whence $q \equiv \Diamond q \equiv \Box q$, and thus entail a collapse of *DML* to propositional or "Fregean" logic (cf. Suszko (1968, pp. 11–12)). Now if one insisted on this point, we could remove Axioms 4 and 5 and reintroduce their necessitations as special extra-logical postulates; or we could introduce a set $\mathbf{Z} = \{z_i : 0 \leq i \leq \omega\}$ of special sentential variables playing the role of conjunctions of state-descriptions, and replace Axioms 4 and 5 by $\Diamond \bigwedge z_i$ and $\Box(z_i \supset p) \vee \Box(z_i \supset \neg p)$, respectively. However, the difference seems to be merely a terminological one, for which reason we shall simply call *DML* a logic.

Some interesting formulae of *DML* (easily provable by Theorem 9) are the following:

(a) $\Box \bigwedge P \equiv \bigwedge_{p \in P} \Box p$;
(b) $\Box(x)px \equiv (x)\Box px$ (the Barcan formula and its converse);
(c) $\Diamond(\bigwedge \boldsymbol{SD} \wedge p) \equiv \neg\Diamond(\bigwedge \boldsymbol{SD} \wedge \neg p) \equiv \Box(\bigwedge \boldsymbol{SD} \supset p)$.

It will be noticed that *DML* is rather weak as a doxastic logic, much weaker, in fact, than contemporary doxastic logics based on possible worlds semantics (e.g., Lenzen (1980)). For example, we cannot prove any formula of the form $D_A p$. One of the few positive facts that may be noted is that Axiom 6 clearly reveals the interdependence of thought-ascriptions (Section 4.4). Despite its weakness, one nevertheless may find *DML* too strong: doesn't Axiom 6 imply the "logical omniscience" of A? For an argument that it does not really do so, we refer to Stalnaker (1976)

5.3. **Some Correspondence Results.** *DML* turns into a stronger and more interesting doxastic logic if some additional restrictions are imposed on the interpretations. This is apparent from the following correspondence theorems:

---

be invalid in this interpretation, the theorem is proved. Cf. Keisler (1971, Ch. 4), on $L\omega_1 0$, and Chellas (1980) on $S5$ and E (the latter is similar to the doxastic fragment of *DML*).

Theorem 10: Axiom $(D_A p \wedge D_A q) \supset D_A(p \wedge q)$ corresponds to the condition that $\boldsymbol{T}_A(\mathbf{s})$ is closed under finite conjunction.[34]

Theorem 11: Axiom $\bigwedge_{p \in P} D_A p \supset D_A \bigwedge P$ ("systemic nature of thought") corresponds to the condition that $\boldsymbol{T}_A(\mathbf{s})$ is closed under *arbitrary* conjunction.[35]

Notice that this axiom in turn implies the doxastic Barcan formula $(x)D_A px \supset D_A(x)px$. Therefore we have the remarkable result that the *Einheitlichkeit* of the soul entails the derivability of the doxastic Barcan formula (in complete axiomatizations)!

Theorem 12: Axiom $D_A p \supset D_A D_A p$ ("self-reflexivity of thought") corresponds to the condition that if $t \in \boldsymbol{T}_A(\mathbf{s})$, then there is a $t' \in \boldsymbol{T}_A(\mathbf{s})$ such that $\sigma(t') = \prod\{\mathbf{s} \in \mathbf{S} : t \in \boldsymbol{T}_A(\mathbf{s})\}$.

To put it more transparently: the axiom "if A thinks that φ then A thinks that A thinks that φ" corresponds to the condition that no thought belongs to the soul unless the soul contains a thought which *says that* this thought belongs to the soul.

However, the *Tractatus* does not contain the informal counterparts of any of these additional semantical postulates or corresponding axioms. The reason is clear: we are here once more dealing with issues which are to be settled by empirical investigation (in this case: psychology), not by logic.[36]

6. Conclusion

This ends our tortuous path through the Tractarian labyrinth. We certainly have not discussed all topics we might have treated: for example, Wittgenstein's views on functions and the theory of types may presumably also be handled by formal means. However, with the above the foundations of formal Tractarian semantics have been laid; in particular, we have achieved our goal of giving simple truth-functional analyses of quantification, the modalities and the propositional attitudes, which is something previous commentators generally considered impossible. This may suffice for a first start.

How does the *Tractatus* look in the light of our formal analysis? From a general point of view, we have obtained a better idea of the general nature of the work: it anticipates Tarski's and Carnap's later work, but it does so in a rather apodictic way. Deriving the consequences of the statements and clarifying their interrelations is a task almost exclusively left to formal analysis. When carrying out the latter, several weaks points emerge. For example, we encountered various inconsistencies; moreover, the work contains various lacunae which must be filled in by our own imagination—just recall the silence of the *Tractatus* on the syntax and semantics of non-elementary pictures and the conspicuously absent answer to TLP 5.54–5.541.

Nevertheless, however crudely it may sometimes have been formulated, the work contains much that is still of interest today. Thus, the quantified logic we have extracted from it is a complete higher-order logic, more comprehensive than standard first-order logic because it treats predicates as names; the doxastic logic we arrived at is perfectly acceptable to contemporary doxastic logicians of the "possible worlds" persuasion; the Tractarian semantics of propositional attitude ascriptions strikingly anticipates modern "language of thought" theories; and the Tractarian

[34]That is, if $t \in \boldsymbol{T}_A(\mathbf{s})$ and $t' \in \boldsymbol{T}_A(\mathbf{s})$, then $(t \wedge t') \in \boldsymbol{T}_A(\mathbf{s})$. (Here "$\wedge$" is a "mental connective" in the sense of Harman (1973).)

[35]The appellation "systemic", which means approximately the same as our *einheitlich*, is due to Routley & Routley (1975). According to the Routleys all thought is systemic. In the next note we shall see that it is not.

[36]For example, the question as to whether the soul is unified can only be settled by empirical research. (Recent research suggests it is not always unified: "split-brain" patients display manifestly non-systemic thoughts, perceptions and memories, as has been noticed by Barwise (1981).)

thesis of the complete describability of the world by elementary sentences is a variant of the doctrine of the supervenience of the mental on the non-mental which is currently coming into vogue.

From a philosophical point of view, the Tractarian philosophy of mind is the most interesting subject we have discussed: here the *Tractatus* manages to combine two (unrelated) theories—the "language of thought" and "supervenience" theories—which certainly do not yet look antiquated today. This is not to say that there are no differences with these modern theories. First, Wittgenstein does not wrestle with the much-debated problem of current "language of thought" theories as to whether thoughts are iconic (picture-like) or discursive (sentence-like), because according to him sentences are iconic too. Secondly, modern psychophysical supervenience theories postulate the supervenience of propositional attitude ascriptions on physical descriptions of the world without imposing further restrictions on these descriptions. The Tractarian theory is more specific in asserting that the physical descriptions in question are, in the final analysis, elementary sentences. Thirdly, according to Wittgenstein meanings are conferred to pictorial elements by the "metaphysical subject", whereas modern theories seek to provide *causal* accounts of the attribution of meanings. To Wittgenstein the latter route is blocked, for he regards believing in causality as a "superstition" (TLP 5.1361, 6.32 ff.).[37] The main weakness of Wittgenstein's account seems to lie in the latter feature, although this may still appeal to some philosophers of a metaphysical bent. However, apart from this feature (which does not belong to logic anyway) the Tractarian theory seems no less attractive and viable than its related modern successors.

All in all, we think our effort has made it clear that the *Tractatus* may fruitfully be discussed in formal terms; the formal approach uncovers various viewpoints which are still interesting in their own right and thereby justifies a greater appreciation of the work than would otherwise be warranted. One may not always agree with the specific form our interpretation has taken: but even in this case a formal account has the advantage over an informal one that it may at least be precisely understood *what* it is one does not agree with. So even if our analysis is not *unantastbar und definitiv*, it may at least facilitate further understanding. *Mögen andere kommen und es besser machen!*

Appendix (1992)

A letter from G. Kreisel (Baden, Switzerland, October 21, 1990; quoted with permission) puts some things we have said in the above in a slightly different light. Kreisel writes as follows about a conversation which he had with Wittgenstein "probably back in 1942":

> (a) It seemed to me too obvious even to mention that Tractatus was concerned with a Boolean algebra; specifically, the algebra generated from the *simples* as elements. Also Wittgenstein would have been horrified at such (for him) pretentious language: one spoke of propositional combinations.
>
> (b) His (only) remark to me was in reply to an observation. I said that what I found in Tractatus was compelling only if one assumed that there were *finitely* many simples. Otherwise things became contrived here and there.
>
> (c) As so often, Wittgenstein seemed (to me) quite unduly pleased with me (and at the time I had no idea, why). He said something to the effect: 'Of course, I thought of the primitive

[37] See Kenny (1984) for an exposition of the Tractarian view, and Field (1978) as a protagonist of the modern approach.

> case, and if things are clear there, the rest will look after itself. If a foundational scheme doesn't work out as simply as it looks, it's no good at all.'
>
> (d) Today I think I know what he liked about (b). It was a straightforward comment without any agonizing. In normal circumstances this would not be much to write home about, but in 'exact philosophy' a modicum of a sense of proportion was a rarity; just think of Carnap's style (or Tarski's in the 30's, not after the 50's). Besides, when it comes to agonizing, few can match Wittgenstein's particular talent for this activity.

In other words, Wittgenstein seems to have been thinking of a situational Boolean algebra which is generated by a finite number of *Sachverhalte*. In such an algebra, there is only a finite number of *Sachlagen* and only a finite number of worlds.

It is hard, if not impossible, to reconcile this view with TLP 4.463, in which it is said that logical space is "infinite".

It is, however, not difficult to modify our reconstruction in the appropriate way: clause (b) of Definitions 3, 7, and 11 should be changed in such a way that it becomes true that $Card(\mathbf{SA}) = Card(\boldsymbol{EL}) = Card(\boldsymbol{ET}) < \aleph_0$. As a result, the first point which is made in footnote 27 becomes relevant, and the whole construction would indeed become "less contrived here and there".

The above has no effect on what is said in Definitions 2, 6 and 10. We shall have to allow for the possibility that some "objects" have certain relations to themselves. Therefore we have to admit proto-*Sachverhalte* (in the sense of Definition 2) like $\mathbf{b} * \mathbf{a} * \mathbf{a}$, $\mathbf{b} * \mathbf{a} * \mathbf{a} * \mathbf{a}$, and so on, and perhaps even $\mathbf{a} * \mathbf{a}$ in case $\mathbf{a}$ is a property which applies to itself.

References

Anscombe, G. E. M. (1959), *An Introduction to Wittgenstein's Tractatus*, Hutchinson, London.

Barwise, J. (1981), 'Scenes and other situations', *The Journal of Philosophy* **78**, 369–397.

Berghel, H. L. (1978), Harman's Tractarian thoughts, *in* Leinfellner, Leinfellner, Berghel & Hübner (1978).

Berghel, H. L., Hübner, A. & Köhler, E., eds (1979), *Proceedings of the Third International Wittgenstein Symposium*, Hölder-Pichler-Tempsky, Vienna.

Black, M. (1964), *A Companion to Wittgenstein's 'Tractatus'*, Cornell University Press, Ithaca, N.Y.

Brentano, F. (1973), *Psychologie vom empirischen Standpunkt, Vol. 1 (1874)*, Meiner, Hamburg.

Carnap, R. (1956), *Meaning and Necessity: A Study in Semantics and Modal Logic*, 2nd edn, University of Chicago Press, Chicago.

Chellas, B. F. (1980), *Modal Logic: An Introduction*, Cambridge University Press, Cambridge.

Chisholm, R. M., Marek, J. C., Blackmore, J. T. & Hübner, A., eds (1985), *Proceedings of the Ninth International Wittgenstein Symposium*, Hölder-Pichler-Tempsky, Vienna.

Clark, R. (1976), 'Old foundations for a logic of perception', *Synthese* **33**, 75–99.

Cocchiarella, N. B. (1984), Philosophical perspectives on quantification in tense and modal logic, *in* D. Gabbay & F. Günthner, eds, 'Handbook of Philosophical Logic, Vol. 2', D. Reidel, Dordrecht.

Copi, I. M. (1958), "Tractatus' 5.542', *Analysis* **18**, 102–104.

Czermak, J. (1979), Ein mengentheoretisches Modell der *Tractatus*-Ontologie, *in* Berghel, Hübner & Köhler (1979).

Favrholdt, D. (1964), *An Interpretation and Critique of Wittgenstein's Tractatus*, Munksgaard, Copenhagen.

Fechner, G. T. (1860), *Elemente der Psychophysik, Vol. 2*, Breitkopf und Härtel, Leipzig.

Field, H. (1978), 'Mental representation', *Erkenntnis* **13**, 9–61.

Fogelin, R. J. (1976), *Wittgenstein*, Routledge and Kegan Paul, London.

Griffin, J. (1965), *Wittgenstein's Logical Atomism*, Oxford University Press, Oxford.

Haller, R. & Grassl, W., eds (1980), *Proceedings of the Fourth International Wittgenstein Symposium*, Hölder-Pichler-Tempsky, Vienna.

Harman, G. (1973), *Thought*, Princeton University Press, Princeton.

Haugeland, J. (1982), 'Weak supervenience', *American Philosophical Quarterly* **19**, 93–103.
Hintikka, J. (1958), 'On Wittgenstein's 'solipsism'', *Mind* **67**, 88–91.
Hintikka, J. & Hintikka, M. B. (1983), 'Some remarks on (Wittgensteinian) logical form', *Synthese* **56**, 155–170.
Hughes, G. E. & Cresswell, M. J. (1972), *An Introduction to Modal Logic*, second printing edn, Methuen, London.
Humberstone, L. (1986), 'Extensionality in sentence position', *Journal of Philosophical Logic* **15**, 27–54.
Keisler, H. J. (1971), *Model Theory for Infinitary Logic*, North-Holland, Amsterdam.
Kenny, A. (1984), *The Legacy of Wittgenstein*, Basil Blackwell, Oxford, chapter Wittgenstein's Early Philosophy of Mind.
Lee, D., ed. (1980), *Wittgenstein's Lectures, Cambridge 1930–1932*, Blackwell, Oxford.
Leinfellner, E., Leinfellner, W., Berghel, H. L. & Hübner, A., eds (1978), *Proceedings of the Second International Wittgenstein Symposium*, Hölder-Pichler-Tempsky, Vienna.
Lenzen, W. (1980), *Glauben, Wissen und Wahrscheinlichkeit*, Springer, Vienna.
Lokhorst, G. J. C. (1985 *a*), A formalization of Wittgenstein's early theory of belief, *in* Chisholm, Marek, Blackmore & Hübner (1985).
Lokhorst, G. J. C. (1985 *b*), Ontologie, semantiek en filosofie van de geest in Wittgensteins Tractatus. Unpublished thesis, Erasmus University, Rotterdam.
MacKay, A. & Merrill, D., eds (1976), *Issues in the Philosophy of Language*, Yale University Press, New Haven.
Menger, K. (1980), Language and mathematics, *in* Haller & Grassl (1980).
Mudersbach, K. (1968), Versuch einer Axiomatisierung der Sätze 1–2.063 des *Tractatus Logico-Philosophicus*. Unpublished.
Mudersbach, K. (1978), A tentative axiomatization of the ontology of the *Tractatus*, *in* Leinfellner et al. (1978).
Perzanowski, J. (1985), Some observations on modal logics and the *Tractatus*, *in* Chisholm et al. (1985).
Routley, R. & Routley, V. (1975), 'The role of inconsistent and incomplete theories in the logic of belief', *Communication and Cognition* **8**, 185–235.
Russell, B. (1918), The philosophy of logical atomism (1918), *in* R. C. Marsh, ed., 'Logic and Knowledge', Allen and Unwin, London.
Russell, B. (1922), Introduction, in *Tractatus Logico-Philosophicus, ed. and tr. by D. F. Pears and B. F. McGuinness* (Wittgenstein 1971).
Russell, B. (1927), Appendix C, in *Principia Mathematica* (Whitehead & Russell 1927).
Russell, B. (1984), *Theory of Knowledge: The 1913 Manuscript, ed. by E. R. Eames and K. Blackwell*, Allen and Unwin, London.
Simons, P. (1986), 'Tractatus Mereologico-Philosophicus?', *Grazer Philosophische Studien* **28**, 165–186.
Skyrms, B. (1981), 'Tractarian nominalism', *Philosophical Studies* **40**, 199–206.
Soames, B. (1983), 'Generality, truth functions, and expressive capacity in the *Tractatus*', *Philosophical Review* **92**, 573–589.
Stalnaker, R. (1976), Propositions, *in* MacKay & Merrill (1976).
Stegmüller, W. (1966), 'Eine modelltheoretische Präzisierung der Wittgensteinschen Bildtheorie', *Notre Dame Journal of Formal Logic* **7**, 181–195.
Suszko, R. (1968), 'Ontology in the *Tractatus* of L. Wittgenstein', *Notre Dame Journal of Formal Logic* **9**, 7–33.
Urmson, J. O. (1956), *Philosophical Analysis*, Basil Blackwell, Oxford.
von Wright, G. H. (1982 *a*), Modal logic and the *Tractatus*, in *Wittgenstein* (von Wright 1982 *b*).
von Wright, G. H. (1982 *b*), *Wittgenstein*, Basil Blackwell, Oxford.
Whitehead, A. N. & Russell, B. (1927), *Principia Mathematica*, 2nd edn, Cambridge University Press, Cambridge.
Wittgenstein, L. (1964), *Philosophische Bemerkungen, ed. by R. Rhees*, Basil Blackwell, Oxford.
Wittgenstein, L. (1971), *Tractatus Logico-Philosophicus, ed. and tr. by D. F. Pears and B. F. McGuinness*, 2nd edn, Routledge and Kegan Paul, London.
Wittgenstein, L. (1979), *Notebooks, ed. and tr. by G. H. Von Wright and G. E. M. Anscombe*, 2nd edn, Basil Blackwell, Oxford.
Wolniewicz, B. (1968), *Rzeczy i fakty*, Panstwowe Wydawnictwo Naukowe, Warsaw.
Wundt, W. (1874), *Grundzüge der physiologischen Psychologie*, Engelmann, Leipzig.